Type Theory – 2017-07-18

Exercise 1. *Prove that, in any cartesian closed category, the following η law holds, for any objects A, B.*

$$\Lambda\,\mathsf{apply}_{A^B} = \mathsf{id}_{A^B}$$

Exercise 2. *Replace the placeholders below so to make the following typing judgments correct, in the Calculus of Constructions.*

$$\begin{array}{ll}
1) & F : * \to * \to *,\ x : (\prod_{\alpha:*} \alpha \to F\alpha\alpha), \\
 & g : (\prod_{\alpha,\beta,\gamma:*} (\beta \to \gamma) \to F\alpha\beta \to F\alpha\gamma) \\
 & \vdash \boxed{?} : (\prod_{\alpha,\beta:*} \alpha \to \beta \to F\alpha\beta) \\
2) & \alpha : *,\ \beta : *,\ A : \alpha \to *,\ B : \beta \to *, \\
 & h : (\prod_{f:\alpha\to\beta} \prod_{a:\alpha} Aa \to B(fa)),\ x : \alpha,\ w : Ax \\
 & \vdash \boxed{?} : \prod_{b:\beta} Bb
\end{array}$$

Exercise 3. *In System F extended with products, consider two types τ, σ with no free occurrence of α. Prove that, in any parametric model, if*

$$f : \forall\alpha.\ (\tau \times \alpha) \to (\tau \times (\alpha \times \sigma))$$

then (informally) for any γ and any $x : \tau, y : \gamma$ we have $\pi_1(\pi_2(f[\gamma](x, y))) = y$. To do so, establish the following formal category-theoretic statement: for any $x : 1 \to \tau, y : 1 \to \gamma$, we have

$$\langle x, y\rangle; f_\gamma; \pi_2; \pi_1 = y$$

Exercise 4. *Consider a diagram D of the form $A \xrightarrow{a} Z \xleftarrow{b} B$. A D-*cone *over such diagram is a triple $(X, f : X \to A, g : X \to B)$ such that $f; a = g; b$.*

1. *Prove that D-cones form a category, where a morphism $(X, f, g) \to (X', f', g')$ is a morphism $h : X \to X'$ such that $f = h; f'$ and $g = h; g'$.*

2. *Prove that, when $Z = 1$ is a final object, there exists a final D-cone if and only if there exists a product object $A \times B$.*

Exercise 5. *Consider an extension of System F with existential types $\exists\alpha.\tau$ (where α can occur in τ) with typing and computational rules*

$$\frac{\Gamma \vdash t : \tau\{\sigma/\alpha\}}{\Gamma \vdash \langle\sigma, t\rangle_{\alpha,\tau} : \exists\alpha.\ \tau}[\exists I] \qquad \frac{\Gamma \vdash t : \exists\alpha.\tau \quad \Gamma, x : \tau \vdash e : \gamma \quad \alpha \notin free(\Gamma, \gamma)}{\Gamma \vdash \mathsf{case}\ t\ \mathsf{of}\ \langle\alpha, x\rangle \to e : \gamma}[\exists E]$$

$$(\mathsf{case}\ \langle\sigma, t\rangle_{\alpha,\tau}\ \mathsf{of}\ \langle\alpha, x\rangle \to e) \to_\beta e\{\sigma/\alpha, t/x\} \qquad (\eta \text{ rule omitted})$$

1. *Assume $\alpha \notin free(\delta)$. Prove there is an isomorphism*

   $$f :\ (\exists\alpha.\ \tau) \to \delta\ \simeq\ (\forall\alpha.\ \tau \to \delta)$$

   by providing two suitable λ terms f, f^{-1}. Verify that $f(f^{-1}(t)) =_{\beta\eta} t$ for any $t : \forall\alpha.\ \tau \to \delta$. (You do not have to verify the other direction.)

2. *Exploit the above property to prove that, if β does not occur in τ, in any parametric model we have the isomorphism*

   $$(\exists\alpha.\ \tau) \simeq (\forall\beta.\ (\forall\alpha.\ \tau \to \beta) \to \beta)$$