Type Theory – 2022-06-27

Exercise 1. *State the Yoneda lemma, in any one of the forms we have seen during the course. Provide a proof for it (or at least a solid sketch).*

Exercise 2. *Replace the placeholders below so to make the following typing judgments correct according to the Calculus of Constructions.*

1) $\alpha : *,\ P : \alpha \to \alpha \to *$
$\vdash \boxed{?} : (\prod_{x:\alpha} \prod_{f:\alpha\to\alpha} Px(fx)) \to (\prod_{y:\alpha} \prod_{g:\alpha\to\alpha} P(gy)y)$

2) $\alpha : *,\ Q : \alpha \to \alpha \to *,\ f : \prod_{x:\alpha} \prod_{y:\alpha} \prod_{P:\alpha\to*} Qxy \to Px \to Py$
$\vdash \boxed{?} : \prod_{a:\alpha} \prod_{b:\alpha} Qab \to Qbb$

Exercise 3. *In a parametric model of System F extended with algebraic types $0, 1, +, \times$, consider the interpretation of the following types. Below, τ, δ are types which do not depend on α.*

$$\forall\alpha.\ (\alpha \times (\delta \to \alpha)) \to (\tau \to \alpha)$$
$$\forall\alpha.\ ((\delta \to \alpha) + ((\alpha + 1) \to 0)) \to \tau$$

For each type above, construct an isomorphic type whose definition involves no quantifiers ($\forall$). Justify your answer.

Exercise 4. *Let $\mathcal{C}$ be a locally small Cartesian closed category.*

1. *Define an isomorphism*

$$\gamma : \mathcal{C}(1, B^A) \cong \mathcal{C}(A, B)$$

   by providing an explicit form for both $\gamma(r)$ and $\gamma^{-1}(s)$. (You do not have to prove that they actually are isomorphisms.)

2. *Prove that, in the context of the simply-typed lambda calculus, we have*

$$\gamma(\llbracket \vdash (\lambda x : \delta.\ x) : \delta \to \delta \rrbracket) = \mathsf{id}_{\llbracket\delta\rrbracket}$$

Exercise 5. *(continues from Exercise 4.)*

1. *Let f, g two simply-typed lambda calculus terms such that*

$$\vdash f : \tau \to \sigma \qquad \vdash g : \sigma \to \tau$$

   Prove that the following morphisms of $\mathcal{C}$ are equal. (Hint: exploit the lemma below, which you can assume to hold.)

$$\gamma(\llbracket \vdash (\lambda x : \tau.\ g\,(f\,x)) : \tau \to \tau \rrbracket) = \gamma(\llbracket \vdash f : \tau \to \sigma \rrbracket)\ ;\ \gamma(\llbracket \vdash g : \sigma \to \tau \rrbracket)$$

2. *Combine all the results above (including those from Exercise 4.) to prove*

$$g(f\,x) =_\beta x \ \wedge\ f(g\,y) =_\beta y \implies \llbracket\tau\rrbracket \cong \llbracket\sigma\rrbracket$$

Lemma 1 (semantic weakening)**.** *In the simply-typed lambda calculus, if α, β are types and $\Gamma \vdash h : \beta$ is a judgment, then*

$$\llbracket \Gamma, x : \alpha \vdash h : \beta \rrbracket \quad = \quad \pi_1\ ;\ \llbracket \Gamma \vdash h : \beta \rrbracket$$