Type Theory – 2020-06-16

Exercise 1. *Let $\mathcal{C}$ be a category, and $F : \mathcal{C} \to \mathcal{C}$ be a functor. Prove that, if $(A, a : FA \to A)$ is an initial F-algebra, then a is an isomorphism.*

Exercise 2. *Replace the placeholders below so to make the following typing judgments correct according to the Calculus of Constructions.*

1) $C : * \to * \to *,\ f : \prod_{\alpha:*} \prod_{\beta:*} \prod_{a:\alpha} \prod_{b:\beta} C\alpha\beta,$
$\tau : *,\ \sigma : *,\ g : (C\tau(\tau \to \tau)) \to \sigma$
$\vdash \boxed{?} : \tau \to \sigma$

2) $P : * \to *,\ Q : * \to *,\ g : \prod_{\beta:*} \prod_{\gamma:*} (\beta \to \gamma) \to P\beta \to Q\gamma,$
$\tau : *,\ x : \prod_{\alpha:*} (\tau \to \alpha) \to P\alpha \vdash \boxed{?} : Q\tau$

Exercise 3. *Consider the following typing judgments in the simply-typed λ calculus. Show how to interpret them in a cartesian closed category $\mathcal{C}$. Below, τ, σ are basic types.*

$$f : \tau \to (\sigma \times \tau) \vdash (\lambda x : \tau.\ \pi_1(fx)) : \tau \to \sigma$$
$$g : ((\tau \to \tau) \times \tau) \to \sigma,\ x : \tau \vdash g\langle(\lambda y : \tau.\ x),\ x\rangle : \sigma$$

Exercise 4. *Consider the category $Set^2 = Set \times Set$.*

1. *Prove that Set^2 has a final object $1_{Set^2} = (1_{Set}, 1_{Set})$.*
2. *Prove that Set^2 has binary coproducts, where*
$$(A, B) +_{Set^2} (A', B') = (A +_{Set} A', B +_{Set} B')$$
3. *Prove that, in Set^2 we have exactly* four *morphisms $1 \to 1 + 1$.*

Exercise 5. *Let $\mathcal{C}$ and $\mathcal{D}$ be two locally small categories, and $F : \mathcal{C} \to \mathcal{D}$ be a functor.*

1. *Consider the following functors $\mathcal{C}^{op} \times \mathcal{C} \to Set$:*
$$G(A, B) = \mathcal{C}(A, B) \qquad H(A, B) = \mathcal{D}(FA, FB)$$
Describe how G and H act on morphisms of $\mathcal{C}^{op} \times \mathcal{C}$.
2. *Consider the following family of functions, indexed over $A, B \in |\mathcal{C}|$:*
$$\eta_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$$
$$\eta_{A,B}(f) = Ff$$
Prove that η is a natural transformation $\eta : G \to H$.