Type Theory – 2018-06-27

Exercise 1. *State the Yoneda lemma, in any one of the forms seen during the course. Provide a proof for it (or at least a solid sketch).*

Exercise 2. *Replace the placeholders below so to make the following typing judgments correct according to the Calculus of Constructions.*

1) $\alpha : *,\ \beta : *,\ h : \prod_{\gamma:*} \prod_{\delta:*} \boxed{?} \to \boxed{?} \to \boxed{?}$
$\vdash h\alpha\beta \boxed{?} : \alpha \to \beta$

2) $\alpha : *,\ a : \alpha,\ R : \alpha \to \alpha \to *,\ s : \prod_{x:\alpha} \prod_{y:\alpha} Rxy \to Ryx,$
$t : \prod_{x:\alpha} \prod_{y:\alpha} \prod_{z:\alpha} Rxy \to Ryz \to Rxz$
$\vdash \boxed{?} : \prod_{x:\alpha} Rax \to Rxx$

Exercise 3. *Consider the interpretation of System F in a parametric model. Let τ be a type not depending on type variable α.*

1. *Define $\sigma(\alpha) \equiv (\tau \to \alpha)$, and verify that it is functorial on α by providing a λ-term of type $\forall \beta, \gamma.\ (\beta \to \gamma) \to \sigma(\beta) \to \sigma(\gamma)$. (Omit the verification of the functor laws.)*

2. *Prove that, in the model, the type $\forall \alpha.((\tau \to \alpha) \times \tau) \to \alpha$ is isomorphic to $\tau \to \tau$.*

Exercise 4. *Consider the standard interpretation of the simply-typed λ calculus in a cartesian closed category $\mathcal{C}$. Provide a definition for the morphisms in $\mathcal{C}$ associated to the following typing judgments, where τ, σ are basic types.*

$$f : (\tau \to \tau) \to \sigma \vdash f(\lambda y : \tau.\ y) : \sigma$$

$$f : (\tau \to \tau) \to \sigma,\ g : \tau \to \tau \vdash f(\lambda y : \tau.\ g(g\,y)) : \sigma$$

Exercise 5. *Let $\mathcal{C}$ a category with binary products and coproducts, with two endofunctors $F, G : \mathcal{C} \to \mathcal{C}$. Define the functors $H, L : \mathcal{C} \to \mathcal{C}$ as $HX = FX \times GX$ and $LX = FX + GX$.*

Assume that $(A, a : HA \to A)$ is an initial H-algebra, $(B, b : LB \to B)$ is an initial L-algebra, and $(C, c : FC \to C)$ is an initial F-algebra.

Prove that there exists a morphism $f : A \to C$ such that f can be expressed in the form $\ldots; Hf; \ldots$ where the dots do not depend on f.

Prove that there exists a morphism $g : C \to B$ such that g can be expressed in the form $\ldots; Fg; \ldots$ where the dots do not depend on g.