Computability Final Test — 2014-07-07

Notes.

- Answer both theory questions, and choose and solve two exercises, only. Solving more exercises results in the failure of the test.
- To pass the exam you need to provide a reasonable contribution in *both* Theory and Exercises parts.
- Exercises with higher number award more points. To achieve a score ≥ 28 you have to solve an exercise marked with $\star$ below.
- Significantly wrong answers will result in negative scores.
- Always provide a justification for your answers.

Reminder: when equating the results of partial functions (as in $\phi_i(3) = \phi_i(5)$), we mean that either 1) both sides of the equation are defined, and evaluate to the same natural number, or 2) both sides are undefined.

Theory

Question 1. *Prove the Rice theorem.*

Question 2. *Prove that $A \in \mathcal{R}$ if and only if $A, \bar{A} \in \mathcal{RE}$.*

Exercises

Exercise 3. *Let f be an arbitrary total recursive function. Prove whether*

$$A = \{n \mid \exists x \in \mathbb{N}.\ f(x) < n \land \phi_n(x+1) = f(x)\} \in \mathcal{RE}$$

Solution (sketch). Consider the property

$$p(x,n) = \text{“}f(x) < n \land \phi_n(x+1) = f(x)\text{”}$$

We have that $p \in \mathcal{RE}$ since we can write a semiverifier as follows:

$S_p(x,n)$:
 $z := f(x)$
 run $\phi_n(x+1)$ and take its result w
 if $(z < n \land w = z)$ then
  return 1
 else
  loop forever

Let us check that this is indeed a semiverifier. First, note that f being total implies that $z := f(x)$ will always halt, and that z will be equal to $f(x)$.

If $p(x,n)$ holds, then $f(x) < n$ and $\phi_n(x+1) = f(x)$, hence $\phi_n(x+1)$ is defined since $f(x)$ is such. So, running $\phi_n(x+1)$ will halt returning a value w

equal to $f(x)$. Hence, the program will reach the if line, and the if guard will evaluate to true, causing S_p to return 1, as it should.

If $p(x, n)$ does not hold, then either $f(x) \geq n$ (since f is total we do not need to consider the case in which $f(x)$ diverges) or $\phi_n(x+1) \neq f(x)$. We consider two cases. 1) If $\phi_n(x+1)$ is undefined, S_p will diverge when evaluating that (as it should). 2) If $\phi_n(x+1)$ is defined, we have two sub-cases: either it returns $w \neq f(x)$ or $f(x) \geq n$. In both sub-cases, the if guard will evaluate to false, making S_p diverge in the last line.

So, we conclude that $p \in \mathcal{RE}$. Since A is defined adding an existential quantification on top of p, A is also $\mathcal{RE}$. □

Exercise 4. *Prove whether*

$$B = \{n \mid \exists x \in \mathbb{N}.\ n \cdot \phi_n(x) = 42\} \in \mathcal{R}$$

Solution (sketch). We have $B \subseteq \{0, \ldots, 42\}$, since if $n > 42$ we have that $n \cdot \phi_n(x)$ is either undefined, 0, or $\geq n > 42$. This is because $\phi_n(x)$ is either undefined, 0, or ≥ 1.

So, B is finite, hence recursive. □

Exercise 5. *Below, a total function $f \in \mathbb{N} \to \mathbb{N}$ is given. Prove that $f \notin \mathcal{R}$.*

$$f(n) = \begin{cases} \phi_n(n) & \textit{if } n \in \mathsf{K} \\ 0 & \textit{otherwise} \end{cases}$$

Hint: also consider $h(n) = \#(\lambda x.\ \tilde{\chi}_{\mathsf{K}}(n))$.

Solution (sketch). By contradiction, assume $f \in \mathcal{R}$. Function h is well defined ($\tilde{\chi}_{\mathsf{K}}$ being recursive partial), so h is a (total) recursive function by the s-m-n theorem.

Since both f and h are recursive, $f \circ h$ is also such. We have:

$$\begin{aligned} f(h(n)) &= \begin{cases} \phi_{h(n)}(h(n)) & \text{if } h(n) \in \mathsf{K} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \phi_{h(n)}(h(n)) & \text{if } \phi_{h(n)}(h(n)) \text{ is defined} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \tilde{\chi}_{\mathsf{K}}(n) & \text{if } \tilde{\chi}_{\mathsf{K}}(n) \text{ is defined} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } n \in \mathsf{K} \\ 0 & \text{otherwise} \end{cases} \\ &= \chi_{\mathsf{K}}(n) \end{aligned}$$

Hence, $\mathsf{K} \in \mathcal{R}$ – a contradiction. □

Exercise 6. $\star$ *[5% score] Prove whether it is possible to construct $f \in (\mathbb{N} \rightsquigarrow \mathbb{N})$ such that $f \in \mathcal{R}$, $f \circ f \notin \mathcal{R}$, and $f \circ f \circ f \in \mathcal{R}$. [95% score] Then, prove whether it is possible to construct $f \in (\mathbb{N} \rightsquigarrow \mathbb{N})$ such that $f \notin \mathcal{R}$, $f \circ f \in \mathcal{R}$, and $f \circ f \circ f \notin \mathcal{R}$. (*Hint: *one could look for functions also satisfying $f \circ f = \mathsf{id}$).*

Solution (sketch). Intentionally omitted. □