Computability Assignment
Year 2013/14 - Number 10

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1 Preliminaries

Recall that for $a, b \in \mathbb{N}$, $min\{a, b\}$ is the least element between a and b. Recall also that a set $C \subseteq \mathbb{N}$ is called *upward closed* iff $\forall x \in C.\ \forall y \in \mathbb{N}\ (y > x \implies y \in C)$.

2 Question

Let $g,\ h \in \mathcal{R}$, and define

$$f(x) = \begin{cases} g(x) & \textit{whenever } x \in \mathsf{K} \\ \min\{g(x), h(x)\} + 1 & \textit{otherwise} \end{cases}$$

Is it possible to find g and h such that $f \in \mathcal{R}$ and total? If it is so, provide g, h, and the proof that $f \in \mathcal{R}$ and total; otherwise, provide a proof of why $f \notin \mathcal{R}$ or not total regardless of the choice of g and h.

2.1 Answer

We can claim that f can be recursive if the body of function f is recursive by the if-then-else lemma, that means:

1. both g and h are recursive;
2. operator $+$ and constant 1 are recursive;
3. the conditions are recursive.

Surely point 2 is true, and we can easily find two function g,$h \in \mathcal{R}$. But f cannot be recursive since point 3: in fact the condition $x \in \mathsf{K}$ is out of $\mathcal{R}$.

Let's proceed by contraddiction: if the condition was recursive, than we could found a verifier as the following one:

$V_A(x)$:

r := $\phi_x(x)$

if r is undefined, then return $\min\{g(x), h(x)\} + 1$

else return $g(x)$

But if a such verifier does exist, then it would be true that also exists a verifier V_{K} for K, and we all know that this is impossible, so we have reached a contraddiction meaning that f cannot be recursive.

3 Question

Prove or disprove: there exists an upward closed set $C \notin \mathcal{RE}$.

3.1 Answer

In practice we have an infinite set $C = [m \ldots \infty) = \{n | n \geq m\}$. This set is surely recursively enumerable because we can build its characteristic function:

$f(n) = \begin{cases} 1 & \text{if } n \geq m \\ undefined & otherwise \end{cases}$. By the if-then-else lemma, $f \in \mathcal{R} \implies C \in \mathcal{RE}$.

4 Question

Prove or disprove: the function f defined below belongs to $\mathcal{R}$.

$$f(n) = \begin{cases} (\varphi_n(n))^n & whenever\ \varphi_n(n)\ is\ defined \\ 77 & otherwise \end{cases}$$

4.1 Answer

No because:

$f(n) = \begin{cases} (\varphi_n(n))^n & whenever\ n \in \mathsf{K} \\ 77 & otherwise \end{cases}$

and by the if-then-else lemma this function is not recursive (but recursively enumerable) because $\mathsf{K} \notin \mathcal{R}$ (but $\mathsf{K} \in \mathcal{RE}$), so $\mathsf{K} \notin \mathcal{R} \implies f \notin \mathcal{R}$.