Year 2013/14 - Number 10

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Preliminaries 1. Recall that for $a, b \in \mathbb{N}, min\{a, b\}$ is the least element between a and b. Recall also that a set $C \subseteq \mathbb{N}$ is called *upward closed* iff $\forall x \in C.\forall y \in \mathbb{N}(y > x \Rightarrow y \in C)$.

Question 2. Let $g, h \in \mathcal{R}$, and define

$$f(x) = \begin{cases} g(x) & \textit{whenever } x \in \mathcal{K} \\ min\{g(x), h(x)\} + 1 & \textit{otherwise} \end{cases}$$

Is it possible to find g and h such that $f \in \mathcal{R}$ and total? If it is so, provide g, h, and the proof that $f \in \mathcal{R}$ and total; otherwise, provide a proof of why $f \notin \mathcal{R}$ or not total regardless of the choice of g and h.

Answer 2.1. We claim that f cannot be recursive and total regardless of the choice of g and h. By contradiction, suppose $f \in \mathcal{R}$ total. Consider the set $A = \{n | f(n) = min\{g(n), h(n)\} + 1\}$ and notice that f is total iff g is total, since $f(n)$ depends on $g(n)$ for all $n \in \mathbb{N}$. We claim that $A \in \mathcal{R}$, since we are able to build a verifier for A:
$V_A(n)$: *compute* $f(n) = x$; *compute* $g(n) = y$; *if* $x = y$ *then return 0* *else return 1*
(Pay attention: it is always possible to compute $f(n)$ and $g(n)$ since they are recursive total functions. Furthermore $f(n) = g(n)$ iff $n \in \mathcal{K}$ because if $n \notin \mathcal{K}$ then $f(n) = g(n) + 1$ or $f(n) = h(n) + 1$.)
We conclude that $A \in \mathcal{R}$. Since $A = \{n | n \notin \mathcal{K}\} = \overline{\mathcal{K}}$, it follows that it should be $\overline{\mathcal{K}} \in \mathcal{R}$, which is impossible because we have already proved that this set is not recursive. We reached a contradiction, therefore $f \notin \mathcal{R}$ total.
It is possible to find g and h such that $f \in \mathcal{R}$ or such that f is total:

- $f \in \mathcal{R}$: pick $g \in \mathcal{R}$ total and $h(x) = undefined$ for all $x \in \mathbb{N}$. Then

$$f(x) = \begin{cases} g(x) & \textit{whenever } x \in \mathcal{K} \\ undefined & \textit{otherwise} \end{cases}$$

  By the if-then-else lemma on $\mathcal{RE}$ sets $f \in \mathcal{R}$ (g is recursive, $undefined$ is recursive and $\mathcal{K} \in \mathcal{RE}$, therefore I can apply the lemma).

- f total: pick $g, h \in \mathcal{R}$ total.

Question 3. Prove or disprove: there exists an upward closed set $C \notin \mathcal{RE}$.

Answer 3.1. Let C be a non-empty upward closed set ($C = \emptyset$ is clearly a $\mathcal{RE}$ set). Consider the function

$$f(n) = \begin{cases} 1 & \textit{if } n \geq k \\ undefined & \textit{otherwise} \end{cases}$$

We claim that f is a characteristic function of C, since we have already proved that a upward closed set C can be seen as $C = \{n | n \geq k\}$ for some $k \in \mathbb{N}$. By the if-then-else lemma on $\mathcal{RE}$ sets, $f \in \mathcal{R} \Rightarrow C \in \mathcal{RE}$. We conclude that $\nexists C$ upward set such that $C \notin \mathcal{RE}$.

Question 4. Prove or disprove: the function f defined below belongs to $\mathcal{R}$.

$$f(n) = \begin{cases} (\varphi_n(n))^n & \textit{whenever } \varphi_n(n) \textit{is defined} \\ 77 & \textit{otherwise} \end{cases}$$

Answer 4.1. We claim that $f \notin \mathcal{R}$. By contradiction, suppose that $f \in \mathcal{R}$. Therefore the set $A = \{n | f(n) = 77\} \in \mathcal{R}$, since we are able to build a verifier for A:
$V_A(n):$ *compute* $f(n) = z$ *if* $z = 77$ *then return 1* *else retu*
(Pay attention: it is always possible to compute $f(n)$ because $f \in \mathcal{R}$ total.)
Notice that $A = \{n | \varphi_n(n) \textit{ undefined}\} = \overline{\mathcal{K}}$. Since $A \in \mathcal{R}$, it should be $\overline{\mathcal{K}} \in \mathcal{R} \rightsquigarrow$ CONTRADICTION! We already know that $\overline{\mathcal{K}} \notin \mathcal{R}$. We conclude that $f \notin \mathcal{R}$.