Computability Assignment
Year 2013/14 - Number 9

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1 Definition

If $\sim$ is an equivalence relation over a set A, a set $B \subseteq A$ is closed under $\sim$ if $\forall x \in B\ \forall y \in A\ (y \sim x \Rightarrow y \in B)$.

2 Question

Let $\sim$ be the relation over $\mathbb{N}$ defined as $x \sim y$ if $|x - y|$ is a multiple of 3. Show that $\sim$ is an equivalence relation and determine all sets of natural numbers closed under $\sim$.

Hint 1: there is only a finite number of such sets.

Hint 2: take a look at question 3 below.

2.1 Answer

In order to show that $\sim$ is an equivalence relation, we have to check that the properties of reflexivity, simmetry and transitivity are satisfied:

1. reflexivity: $x \sim x$ if $|x - x| = 3k, k \in \mathbb{N}$, which is true since $0 = 3 \times 0$.
2. simmetry: if $x \sim y$ then $y \sim x$. But$|x - y| = 3k, k \in \mathbb{N}$, then $|y - x| = 3k, k \in \mathbb{N}$, since $|x - y| = |y - x|$.
3. transitivity: if $x \sim y$ and $y \sim z$ then $x \sim z$. But if $x \sim y$ then $|x - y| = 3k, k \in \mathbb{N}$, if $y \sim z$ then $|y - z| = 3h, h \in \mathbb{N}$. Now, $|x - z| = |(x - y) + (y - z)|$ and we know that $|x - y|$ and $|y - z|$ are multiples of 3. This means that $x - y = \pm 3n$, and $y - z = \pm 3m$, thus $|x - y| = |3(\pm n \pm m)| = 3|\pm n \pm m| = 3k, k \in \mathbb{N}$.

There are three equivalence classes of elements of $\mathbb{N}$: $[0], [1], [2]$. For $n \in \mathbb{N}$, we have that either $|n - 0| = 3m$ for some $m \in \mathbb{N}$, thus $n \in [0]$, or $|n - 0| \neq 3m$ for $m \in \mathbb{N}$, thus we can only have $|n - 0| = 3m + 1$ or $|n - 0| = 3m + 2$ (otherwise n would have been a mutiple of 3). In the first case we have that $n = 3m + 1$, thus $n - 1 = 3m$, thus $|n - 1| = 3m$, thus $n \in [1]$, in the second case $n = 3m + 2$, thus $n - 2 = 3m$, thus $|n - 2| = 3m$, thus $n \in [2]$. From (3.1) we know that a set B of natural numbers is closed under $\sim$ iff it is the (possibly empty) union of equivalence classes of elements of A. The set of such equivalence classes on which the union is computed is a subset of $\{[0], [1], [2]\}$, thus the number of possible sets is finite, and equal to $2^3 = 8$.

3 Question

Let $\sim$ be an equivalence relation over a nonempty set A. Prove that a subset $B \subseteq A$ is closed under $\sim$ if and only if it is a (possibly empty) union of equivalence classes of elements of A (for the definition of equivalence class of an element of A, see point 1 of assignment 8).

3.1 Answer

If $B = \emptyset$, thus the empty union of equivalence classes of elements of A, then the statement is trivially true, because for all the elements of A (which is nonempty), there is no $x \in B$ such that $y \sim x$, thus the implication $y \sim x \implies y \in B$ is true because the premise is false, and, on the other hand, if $\forall x \in B\ \forall y \in A\ (y \sim x \Rightarrow y \in B)$ is verified because $B = \emptyset$, then B can be seen as the empty intersection of equivalence classes of elements of A. If $B \neq \emptyset$, then if $B \subseteq A$ is closed under $\sim$, then $\forall x \in B.\forall y \in A.(y \sim x \implies y \in B)$. By definition of equivalence class, $[x] = \{y | y \in A \wedge x \sim y\}$. By contradiction, if we suppose that B is not the union of equivalence classes of elements of A, then there exists at least one element t such that $t \in [x].x \in B$, but $t \in A \backslash B$. But $t \in [x] \implies x \sim t \implies t \sim x$, but, since B is closed under $\sim$, since $\forall x \in B.\forall y \in A.(y \sim x \implies y \in B)$, we have that $(t \sim x \wedge t \in A \wedge x \in B) \implies t \in B$, a contradiction. On the other hand, if B is the union of equivalence classes of elements of A, then if $x \in B$, then $x \in [h]$ for a certain $h \in A$, and since $[h] = \{y | y \in A \wedge h \sim y\}$, all the elements $y \in A$ such that $y \sim h$ (and therefore such that $h \sim y$) also belong to the equivalence class $[h]$, which means that they belong to B, which is therefore closed under $\sim$. They are all the sets B such that $B = \bigcup_{a \in J} a$ with $J \subseteq \{0, 1, 2\}$.