# Year 2013/14 - Number 9

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**Definition 1.** If $\sim$ is an equivalence relation over a set $A$, a set $B \subseteq A$ is closed under $\sim$ if $\forall x \in B \forall y \in A (y \sim x \Rightarrow y \in B)$.

**Question 2.** Let $\sim$ be the relation over $\mathbb{N}$ defined as $x \sim y$ if $|x - y|$ is a multiple of 3. Show that $\sim$ is an equivalence relation and determine all sets of natural numbers closed under $\sim$.
Hint 1: there is only a finite number of such sets.
Hint 2: take a lok at question 3 below.

**Answer 2.1.** Let's prove that $\sim$ is an equivalence relation:

- Reflexivity: $x \sim x \Leftrightarrow |x - x| = 3k$ for some $k \in \mathbb{N}$. The property holds (pick $k = 0$), so the relation is reflexive.
- Symmetry: $x \sim y \Rightarrow |x - y| = 2k$ for some $k \in \mathbb{N}$ $\Rightarrow |(-)(y - x)| = 3k \Rightarrow |(-)||y - x| = 3k \Rightarrow |y - x| = 3k \Rightarrow y \sim x$.
- Transitivity: By hypothesis $x \sim y \Rightarrow |x - y| = 3m$ for some $m \in \mathbb{N}$ and $y \sim z \Rightarrow |y - z| = 3n$ for some $n \in \mathbb{N}$. Consider $|x - z|$. $|x - z| = |(x - y) + (y - z)|$. Since $|x - y| = 3m$, then $(x - y) = \pm 3m$ and since $|y - z| = 3n$, then $(y - z) = \pm 3n$. It follows that $|x - z| = |\pm 3m \pm 3n| = 3| \pm m \pm n|$ where $| \pm m \pm n| = k \in \mathbb{N}$. Since $|x - z| = 3k$, $x \sim z$.

We want to determine al sets of natural numbers closed under $\sim$. First of all, we claim that it exists only 3 equivalence classes of elements of $\mathbb{N}$: [0], [1], [2]. In fact, let $x \in \mathbb{N}$, then we can distinguish two cases:

- $|n - 0| = 3k$ for some $k \in \mathbb{N}$: it follows that $n \in [0]$.

- $\forall k \in \mathbb{N}, |n-0| \neq 3k$: it follows that $|n-0| = 3k+l$, for some $k \in \mathbb{N}$ and $l \in \{1,2\}$:

  - if $|n-0| = 3k+1$, then $n = 3k+1$, i.e. $n-1 = 3k$. Since $m \in [1] \Leftrightarrow |m-1| = 3k$ for some $k \in \mathbb{N}$, then $n \in [1]$.
  - if $|n-0| = 3k+2$, then $n = 3k+2$, i.e. $n-2 = 3k$. Since $m \in [2] \Leftrightarrow |m-2| = 3k$ for some $k \in \mathbb{N}$, then $n \in [2]$.

Recalling the Question 3, $B \subseteq \mathbb{N}$ is closed under $\sim$ iff $B = \bigcap_{j\in J}[j]$ where $J \subseteq \mathbb{N} \Rightarrow B = \bigcap_{j\in J}[j]$ where $J \subseteq \{0,1,2\}$. Since there could be only $2^{|\{0,1,2\}|} = 8$ possible sets $J$, there is only a finite number of such closed sets.

**Question 3.** Let $\sim$ be an equivalence relation over a nonempty set $A$. Prove that a subset $B \subseteq A$ is closed under $\sim$ if and only if it is a (possibly empty) union of equivalence classes of elements of $A$ (for the definition of equivalence class of an element of $A$, see point 1 of assignment 8).

**Answer 3.1.** We claim that $B \subseteq A$ is closed under $\sim \Leftrightarrow B = \bigcup_{z\in J}[z]$ where $J \subseteq A$.
Clearly the property holds if $B = \emptyset$, therefore suppose $B \neq \emptyset$.

- PROOF $\Leftarrow$: Let $B = \bigcup_{z\in J}[z]$. Let $x \in B$, then $\exists y \in A$ such that $x \in [y] \subseteq B$. Since $x \sim y$, for all $z \in A$ such that $x \sim z$, it holds $z \sim y$. It follows that $z \in [y] \Rightarrow z \in B$. Therefore $B$ is closed under $\sim$.

- PROOF $\Rightarrow$: Let $B$ be closed under $\sim$. Since $B \subseteq A$, $\forall x \in B\ \exists y_x \in A$ such that $x \sim y_x$ (for example $y_x = x$, since $\sim$ is a reflexive relation). It follows that $x \in [y_x]$. Let's define $J = \{y_x | x \in B\}$, therefore $B \subseteq \bigcup_{j\in J}[j]$.
  We claim that $\bigcup_{j\in J}[j] \subseteq B$: let $z \in [j] \Rightarrow z \sim j$. Since $j \sim x$ for some $x \in B$ (by definition), then $z \sim x$. $B$ is closed under $\sim$, therefore $z \in B$.