Computability Assignment
Year 2013/14 - Number 8

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1 Preliminaries

Recall that an equivalence relation $\sim$ over a set A is a binary relation that satisfies all of the following:

1. $\forall x \in A.\ x \sim x$ (reflexivity);
2. $\forall x, y \in A.\ x \sim y \Rightarrow y \sim x$ (symmetry);
3. $\forall x, y, z \in A.\ x \sim y \ \wedge \ y \sim z \Rightarrow x \sim z$ (transitivity).

If A is a set and $\sim$ is an equivalence relation over A, then for all $x \in A$ one can define the *equivalence class* of x with respect to $\sim$, that is the set $[x] = \{y | y \in A \ \wedge \ x \sim y\}$. We will denote by $A/{\sim}$ the set of all equivalence classes of elements of A, that is $A/{\sim} = \{[x] | x \in A\}$.

2 Question

Let A be a set and $\sim$ an equivalence relation over A. Show that, for all $x, y \in A$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$. Hint: remember that, by the *law of excluded middle*, for any choice of $x, y \in A$, either $x \sim y$ or $x \not\sim y$ (where $x \not\sim y$ means $\neg(x \sim y)$).

2.1 Answer

By definition, we can say that $[x] = \{z_x | z_x \in A \ \wedge \ x \sim z_x\}$ and $[y] = \{z_y | z_y \in A \ \wedge \ y \sim z_y\}$ with $x, y \in A$. At this point we can have three cases:

1. trivially $x = y \implies [y] = [x]$;

2. $x \neq y \wedge y \in [x] \implies x \in [y]$ (because x and y are in symmetric relation) $\implies x \sim y$. Now we can take any $z_x \neq y$: if $x \sim z_x$ and $y \sim x$, then we can apply the transitivity property and finally say that $z_x \sim y$. So, generalizing, we have that $\Big(\forall z_x \in [x] \setminus \{y\}. z_x \sim y\Big) \implies z_x \in [y]$ (and the same reasoning holds $\forall z_y$ wrt. x) $\implies [x] = [y]$.

3. $x \neq y \wedge y \notin [x] \implies x \notin [y] \implies x \not\sim y \implies [x] \neq [y]$ by negation of case 2.

3 Question

Let $f \in (\mathbb{N} \to \mathbb{N})$. For each of the relations below, prove whether it is an equivalence relation over $\mathbb{N}$:

1. $x \sim y$ if and only if $f(x) = f(y)$;
2. $x \sim y$ if and only if $f(x) \neq f(y)$;
3. $x \sim y$ if and only if $f^{-1}(x) \cap f^{-1}(y) \neq \emptyset$.

3.1 Answer

1. Yes because all the three basic properties hold. That is:
   (a) $\forall x \in \mathbb{N}.\ f(x) = f(x)$ (reflexivity);
   (b) $\forall x, y \in \mathbb{N}.\ f(x) = f(y) \Rightarrow f(y) = f(x)$ (symmetry);
   (c) $\forall x, y, z \in A.\ f(x) = f(y) \wedge f(y) = f(z) \Rightarrow f(x) = f(z)$ (transitivity).
2. No because both properties (a) and (c) does not hold. In fact, for instance, we could have $f(x) \neq f(y)$ and $f(y) \neq f(z)$ but nothing forces us to do not have $f(x) = f(z)$.
3. ...

4 Question

Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be an enumeration for the set of recursive partial functions from $\mathbb{N}$ to $\mathbb{N}$, and let $\sim$ be the equivalence relation over $\mathbb{N}$ defined as follows: $i \sim j$ if and only if $\varphi_i = \varphi_j$. Moreover, let $e \in (\mathbb{N} \times \mathbb{N} \rightsquigarrow \mathbb{N})$ the partial function defined as $e(a, b) = \varphi_a(b)$.

Prove that, if $i \sim j$, then $\forall b \in \mathbb{N},\ e(i, b) = e(j, b)$.

4.1 Answer

An immediate reasoning can simply be the following: $i \sim j \implies \varphi_i = \varphi_j \implies \forall x \in \mathbb{N}. \varphi_i(x) = \varphi_j(x) \implies \forall x \in \mathbb{N}.\ e(i, x) = e(j, x)$.

5 Remark

Notice that, by what you have proved in the previous exercise, it can be deduced that one can obtain a well-defined partial function $f \in (\mathbb{N}/\!\sim \times \mathbb{N} \rightsquigarrow \mathbb{N})$ by posing $f([a], b) = e(a, b)$.