Computability Assignment
Year 2013/14 - Number 8

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1 Preliminaries

Recall that an equivalence relation $\sim$ over a set A is a binary relation that satisfies all of the following:

1. $\forall x \in A.\ x \sim x$ (reflexivity);
2. $\forall x, y \in A.\ x \sim y \Rightarrow y \sim x$ (symmetry);
3. $\forall x, y, z \in A.\ x \sim y \ \wedge\ y \sim z \Rightarrow x \sim z$ (transitivity).

If A is a set and $\sim$ is an equivalence relation over A, then for all $x \in A$ one can define the *equivalence class* of x with respect to $\sim$, that is the set $[x] = \{y | y \in A \ \wedge\ x \sim y\}$. We will denote by $A/\sim$ the set of all equivalence classes of elements of A, that is $A/\sim\, = \{[x] | x \in A\}$.

2 Question

Let A be a set and $\sim$ an equivalence relation over A. Show that, for all $x, y \in A$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$. Hint: remember that, by the *law of excluded middle*, for any choice of x, $y \in A$, either $x \sim y$ or $x \not\sim y$ (where $x \not\sim y$ means $\neg(x \sim y)$).

2.1 Answer

By the law of excluded middle, we have two possible cases: $x \sim y$ and $x \nsim y$. In the first case, we have that, $z \in [x]$ implies $z \in A \wedge x \sim z$. But because by simmetry $y \sim x$, this gives us, by transitivity, that $y \sim z$, which, together with

$z \in A$, implies that $z \in [y]$. We can demonstrate with a similar reasoning that $z \in [y] \implies z \in [x]$, and that gives us that $[x] = [y]$, since $\forall z \in A. z \in [x] \iff z \in [y]$. In the second case, we have that if $z \in [x]$, then $z \in A \wedge x \sim z$. But, then, by contradiction, if $z \in [y]$ we would have that $y \sim z$, which would give us, by simmetry and transitivity, $x \sim y$, contradiction. This means that it has to be $z \in [x] \implies z \notin [y]$. With a similar argument we can demonstrate that $z \in [y] \implies z \notin [x]$ as well, wich means that $x \sim y$ implies $[x] \cap [y] = \emptyset$.

3 Question

Let $f \in (\mathbb{N} \to \mathbb{N})$. For each of the relations below, prove whether it is an equivalence relation over $\mathbb{N}$:

1. $x \sim y$ if and only if $f(x) = f(y)$;
2. $x \sim y$ if and only if $f(x) \neq f(y)$;
3. $x \sim y$ if and only if $f^{-1}(x) \cap f^{-1}(y) \neq \emptyset$.

3.1 Answer

1. Yes, because it satisfies all three properties: reflexivity, because $x \sim x$ iff $f(x) = f(x)$, always true; simmetry, because $x \sim y$ iff $f(x) = f(y)$, but if $f(x) = f(y)$ it is also true that $f(y) = f(x)$, therefore $y \sim x$; transitivity because if $x \sim y$ then $f(x) = f(y)$, and if $y \sim z$ then $f(y) = f(z)$ but then we have $f(x) = f(y) = f(z)$ which gives us $x \sim z$.

2. No, because it does not satisfy the reflexive property: $x \sim x \implies f(x) \neq f(x)$, a contradiction.

3. Yes, because it satisfies all three properties: reflexivity, because $x \sim x$ iff $f^{-1}(x) \cap f^{-1}(x) \neq \emptyset$, always true because it will always contain at least x; simmetry, because $x \sim y$ iff $f^{-1}(x) \cap f^{-1}(y) \neq \emptyset$, but then $f^{-1}(y) \cap f^{-1}(x) \neq \emptyset$, thus $y \sim x$; transitivity, because $(x \sim y \wedge y \sim z) \implies x \sim z$ iff $(f^{-1}(x) \cap f^{-1}(y) \neq \emptyset \wedge f^{-1}(y) \cap f^{-1}(z) \neq \emptyset) \implies f^{-1}(x) \cap f^{-1}(z) \neq \emptyset$, which is true if $x = y = z$ (it's $a \wedge a \implies a$, with$a = f^{-1}(y) \cap f^{-1}(x) \neq \emptyset$, which we have proven to be always true in the reflexivity case), and otherwise, it remain true since $(f^{-1}(x) \cap f^{-1}(y) \neq \emptyset \wedge f^{-1}(y) \cap f^{-1}(z) \neq \emptyset)$ is always false if (since two natural numbers cannot have two different images, the only way the intersection of the f^{-1}'s would be nonempty would be if they were the f^{-1}'s of the same element., but we are not in the case in which $x = y = z$, so at least one has to be different, but then one of the two conditions in the conjunction would be false, thus making the implication true because the premise would be false).

4 Question

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an enumeration for the set of recursive partial functions from $\mathbb{N}$ to $\mathbb{N}$, and let $\sim$ be the equivalence relation over $\mathbb{N}$ defined as follows: $i \sim j$ if and only if $\varphi_i = \varphi_j$. Moreover, let $e \in (\mathbb{N} \times \mathbb{N} \rightsquigarrow \mathbb{N})$ the partial function defined as $e(a, b) = \varphi_a(b)$.

Prove that, if $i \sim j$, then $\forall b \in \mathbb{N},\ e(i, b) = e(j, b)$.

4.1 Answer

Since $i \sim j \implies \varphi_i = \varphi_j$, we have that $\forall b \in \mathbb{N}. e(i, b) = \varphi_i(b) = \varphi_j(b) = e(j, b)$.

5 Remark

Notice that, by what you have proved in the previous exercise, it can be deduced that one can obtain a well-defined partial function $f \in (\mathbb{N}/\!\sim \times \mathbb{N} \rightsquigarrow \mathbb{N})$ by posing $f([a], b) = e(a, b)$.