Year 2013/14 - Number 8

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Preliminaries 1. Recall that an equivalence relation $\sim$ over a set A is a binary relation that satisfies all of the following:

1. $\forall x \in A.x \sim x$ (reflexivity);
2. $\forall x, y \in A.x \sim y \Rightarrow y \sim x$ (symmetry);
3. $\forall x, y, z \in A.x \sim y \wedge y \sim z \Rightarrow x \sim z$ (transitivity).

If A is a set and $\sim$ is an equivalence relation over A, then for all $x \in A$ one can define the *equivalence class* of x with respect to $\sim$, that is the set $[x] = \{y|y \in A \wedge x \sim y\}$. We will denote by $A/\sim$ the set of all equivalence classes of elements of A, that is $A/\sim = \{[x]|x \in A\}$.

Question 2. Let A be a set and $\sim$ an equivalence relation over A. Show that, for all $x, y \in A$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$. Hint: remember that, by the *law of excluded middle*, for any choice of $x, y \in A$, either $x \sim y$ or $x \nsim y$ (where $x \nsim y$ means $\neg(x \sim y)$).

Answer 2.1. By the law of excluded middle, $\forall x, y \in A \Rightarrow x \sim y \vee x \nsim y$:

1. If $x \sim y$, since $[x] = \{y|y \in A \wedge x \sim y\}$, then $y \in [x]$. Since $\sim$ is a transitive relation, $\forall z \in A.y \sim z \Rightarrow x \sim z$. It follows that $z \in [x]$. Since $z \in [y]$, then $[y] \subseteq [x]$. With the same trick, we can prove that even $[x] \subseteq [y]$, therefore $[x] = [y]$.
2. If $x \neq y$, since $[x] = \{y|y \in A \wedge x \sim y\}$, then $y \notin [x]$. Since $\sim$ is a transitive relation, $x \nsim y \Rightarrow x \nsim z \vee z \nsim y$ for some y. Suppose $z \sim y$, then $x \nsim z$. We can state that $\forall z.z \sim y \Rightarrow z \nsim x \Rightarrow z \notin [x]$, i.e. $\forall z.z \in [y] \Rightarrow z \notin [x]$, therefore $[y] \cap [x] \neq \emptyset$.

Question 3. Let $f \in (\mathbb{N} \to \mathbb{N})$. For each of the relations below, prove wheter it is an equivalence relation over $\mathbb{N}$:

1. $x \sim y$ if and only if $f(x) = f(y)$;
2. $x \sim y$ if and only if $f(x) \neq f(y)$;
3. $x \sim y$ if and only if $f^{-1}(x) \cap f^{-1}(y) \neq \emptyset$.

Answer 3.1. 1.

- Reflexivity: $x \sim x$ iff $f(x) = f(x)$ OK
- Symmetry: $x \sim y \Rightarrow f(x) = f(y) \Rightarrow f(y) = f(x) \Rightarrow y \sim x$ OK
- Transitivity: $x \sim y, y \sim z \Rightarrow f(x) = f(y) \wedge f(y) = f(z) \Rightarrow f(x) = f(z) \Rightarrow x \sim z$ OK

This is an equivalence relation.

2.

- Reflexivity: $x \sim x$ iff $f(x) \neq f(x)$ NO

This isn't an equivalence relation.

3.

- Reflexivity: $x \sim x$ iff $f^{-1}(x) \cap f^{-1}(x) \neq \emptyset$. Since $f^{-1}(x) \cap f^{-1}(x) = f^{-1}(x)$, the property holds iff f is surjective. It follows that, in general, the relation $\sim$ is not a reflexive relation.

Since the relation is not always reflexive, it's not even an equivalence relation.

Question 4. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an enumeration for the set of recursive partial functions from $\mathbb{N}$ to $\mathbb{N}$, and let $\sim$ be the equivalence relation over $\mathbb{N}$ defined as follows: $i \sim j$ if and only if $\varphi_i = \varphi_j$. Moreover, let $e \in (\mathbb{N} \times \mathbb{N} \rightsquigarrow \mathbb{N})$ the partial function defined as $e(a, b) = \varphi_a(b)$.
Prove that, if $i \sim j$, then $\forall b \in \mathbb{N}, e(i, b) = e(j, b)$.

Answer 4.1. $i \sim j \Rightarrow \varphi_i = \varphi_j \Rightarrow \forall b \in dom(\varphi_i) = dom(\varphi_j), \varphi_i(b) = \varphi_j(b)$.
Since $e(a, b) = \varphi_a(b)$, $\forall b \in dom(\varphi_i), e(i, b) = \varphi_i(b) = \varphi_j(b) = e(j, b)$.
$\forall b.b \notin dom(\varphi_i) \Rightarrow b \notin dom(\varphi_j)$. Since $b \notin dom(\varphi_a) \Rightarrow (a, b) \notin dom(e)$, $(i, b) \notin dom(e) \wedge (j, b) \notin dom(e)$, i.e. $e(i, b) = e(j, b) = undefined$.

Remark 5. Notice that, by what you have proved in the previous exercise, it can be deduced that one can obtain a well-defined partial function $f \in (\mathbb{N}/\sim \times \mathbb{N} \rightsquigarrow \mathbb{N})$ by posing $f([a], b) = e(a, b)$.