Computability Assignment
Year 2013/14 - Number 7

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1 Question

A set $C \subseteq \mathbb{N}$ is called *upward closed* iff $\forall x \in C.\ \forall y \in \mathbb{N}\ (y > x \implies y \in C)$.

Provide a characterization of the set $Z = \{X | X \in \mathcal{P}(\mathbb{N}) \wedge X \text{ is upward closed}\}$ (i.e. find a property p such that $Z = \{X | X \in \mathcal{P}(\mathbb{N}) \wedge p(X)\}$, where p could be a conjunction of many "simpler" properties).

1.1 Answer

p(x) $= \forall y.(y \notin X \vee (y+1) \in X)$

2 Question

A set $X \subseteq \mathbb{N}$ is called *cofinite* iff $\overline{\{X\}}$ is finite.

Prove or refute the statement: "if $X,\ Y \in \mathcal{P}(\mathbb{N})$ are NOT cofinite, then $X \cup Y$ is NOT cofinite".

2.1 Answer

If X (respectively Y) is not cofinite, then $\overline{\{X\}}$ (respectively $\overline{\{Y\}}$) is infinite.

But we can't say anything about the finiteness of $\overline{\{X \cup Y\}} = \overline{\{X\}} \cap \overline{\{Y\}}$ (and hence about the cofiniteness of $\{X \cup Y\}$), so the statement is not valid.

For instance, take X = {a|a is even}, Y = {a|a is odd}. $\overline{X}$ and $\overline{Y}$ are not finite, so X and Y are not cofinite, but $X \cup Y = \mathbb{N}$, so $\overline{X \cup Y} = \emptyset$ is finite, so $X \cup Y$ is cofinite.

3 Question

In what follows, $A \subseteq \mathbb{N}$.

1. Prove that if there exists a bijection $f \in (\mathbb{N} \to A)$, then A is infinite.
2. Can you provide an example of an infinite set A and of a function $f \in (\mathbb{N} \to A)$ which is neither injective nor surjective?

3.1 Answer

1. By contradiction assume A finite, and that there exists a bijection $f \in (\mathbb{N} \to A)$. Now suppose we take B s.t. $|B| = |A| + 1$, $B \subset \mathbb{N}$. So $|f(B)| \leq |A| < |B|$, then there must be two elements $b_1, b_2 \in B$ s.t. $f(b_1) = f(b_2)$. But then f is not injective, contradicting our hypothesis that f was bijective. So A must be infinite.
2. Take $A = \mathbb{N}$, $f(n) = 0$.