Computability Assignment
Year 2013/14 - Number 7

Please keep this file anonymous: do not write your name inside this file.

More information about assignments at http://disi.unitn.it/~zunino/teaching/computability/assignments

Please do not submit a file containing only the answers; edit this file, instead, filling the answer sections.

1 Question

A set $C \subseteq \mathbb{N}$ is called *upward closed* iff $\forall x \in C.\ \forall y \in \mathbb{N}\ (y > x \Longrightarrow y \in C)$.

Provide a characterization of the set $Z = \{X | X \in \mathcal{P}(\mathbb{N}) \wedge X \text{ is upward closed}\}$ (i.e. find a property p such that $Z = \{X | X \in \mathcal{P}(\mathbb{N}) \wedge p(X)\}$, where p could be a conjunction of many "simpler" properties).

1.1 Answer

$Z = \{X | X \in \mathcal{P}(\mathbb{N}) \wedge \forall x \in X. \forall y \in \mathbb{N}. (y > x \implies y \in C)\}$

2 Question

A set $X \subseteq \mathbb{N}$ is called *cofinite* iff $\overline{\{X\}}$ is finite.

Prove or refute the statement: "if $X,\ Y \in \mathcal{P}(\mathbb{N})$ are NOT cofinite, then $X \cup Y$ is NOT cofinite".

2.1 Answer

If X, Y are not cofinite, then $\mathbb{N} - X$ and $\mathbb{N} - Y$ are infinite. To refute the statement, one has only to show a pair of sets X, Y such that $\overline{X}, \overline{Y}$ are infinite and such that $\overline{X \cup Y}$ is finite. One can take X to be the all even natural numbers, $\overline{X}$ would be all odd natural numbers, and Y to be all odd natural numbers, while $\overline{Y}$ would be all even natural numbers: their union would be $\mathbb{N}$, and $\overline{X \cup Y}$ would be $\emptyset$.

3 Question

In what follows, $A \subseteq \mathbb{N}$.

1. Prove that if there exists a bijection $f \in (\mathbb{N} \to A)$, then A is infinite.

2. Can you provide an example of an infinite set A and of a function $f \in (\mathbb{N} \to A)$ which is neither injective nor surjective?

3.1 Answer

1. By contradiction, suppose that $f \in (\mathbb{N} \to A)$ is a bijective function, and that A is finite. Without loss generality, we can assume that A has k elements. Because f is a total function, there exists $f(i) \in A, i \in N, 1 \leq i \leq k$, and since f is injective, $\forall 1 \leq i, j \leq k, i \neq j. f(i) \neq f(j)$. This means that $f(i) \in A, i \in N, 1 \leq i \leq k$ are all the k distinct elements of A, which means that $f(k+1)$ would have to exist, and be an element of A (because f is total), but would also have to be one of the $f(i) \in A, i \in N, 1 \leq i \leq k$ distinc elements generate by a $f(j)$ for a $j < k+1$, since A only has k elements. But this too shouldn't be possible, since f was assumed to be injective, thus we reached a contradiction.

2. Take A to be the set of all even natural numbers, and $f(x) = 2$ to be the function from $\mathbb{N}$ to A.