Year 2013/14 - Number 7

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Question 1. A set $C \subseteq \mathbb{N}$ is called *upward closed* iff

$$\forall x \in C. \forall y \in \mathbb{N}(y > x \Longrightarrow y \in C)$$

Provide a characterization of the set $Z = \{X | X \in P(\mathbb{N}) \wedge X \text{ is upward closed}\}$ (i.e. find a property p such that $Z = \{X | X \in P(\mathbb{N}) \wedge p(X)\}$, where p could be a conjuntion of many "simpler" properties).

Answer 1.1. Let's consider the upward-closed property. Notice that

$$\forall x \in C. \forall y \in \mathbb{N}(y > x \Longrightarrow y \in C) \Longrightarrow \forall x \in C, \{y \in \mathbb{N} | y > x\} \subseteq C$$

We distinguish two cases:

1. $C = \emptyset$
2. $C \neq \emptyset$: by the well-ordering principle, C has a minimum, say $c \in C$ (i.e. $\forall x \in C, x \geq c$). We claim that

   $$\forall x \in C, \{n \in \mathbb{N} | n > x\} \subseteq \{n \in \mathbb{N} | n > c\}$$

   in fact, $\forall x \in C$, let $m \in \{n \in \mathbb{N} | n > x\} \Longrightarrow m > x$. Since $x \geq c$, $m > c$, therefore $m \in \{n \in \mathbb{N} | n > c\}$.
   We rewrite the property in this way: $\forall x \in C. \forall y \in \mathbb{N}(y > x \Longrightarrow y \in C) \Leftrightarrow \{n \in \mathbb{N} | n > c\} \subseteq C$, where c is the minimum element of C.

Now we are able to characterize the set Z. We define

$$p(X) \equiv \text{"} X = \emptyset \vee \exists m. X = \{n \in \mathbb{N} | n \geq m\} \text{"}$$

Question 2. A set $X \subseteq \mathbb{N}$ is called *cofinite* iff $\overline{\{X\}}$ is finite.
Prove or refute the statement: "if $X, Y \in P(\mathbb{N})$ are NOT cofinite, then $X \cup Y$ is NOT cofinite".

Answer 2.1. We claim that the implication doesn't hold. In fact, we prove that, given the property "$X, Y \in P(\mathbb{N})$ are NOT cofinite", we shall not conclude neither that $X \cup Y$ is NOT cofinite, nor that $X \cup Y$ is cofinite. Let's consider the two cases:

1. suppose that $X, Y \in P(\mathbb{N})$ are NOT cofinite and $X \cup Y$ is NOT cofinite. $X \cup Y$ is NOT cofinite $\Rightarrow \overline{X \cup Y}$ is NOT finite $\Rightarrow \overline{X} \cap \overline{Y}$ is NOT finite. Since X and Y are NOT cofinite, then $\overline{X}$ and $\overline{Y}$ are NOT finite. It's not true that A, B NOT finite $\Rightarrow A \cap B$ NOT finite. For example, take A as the set of even numbers, and B as the set of odd numbers; then $A \cap B = \emptyset$ is finite.

2. suppose that $X, Y \in P(\mathbb{N})$ are NOT cofinite and $X \cup Y$ is cofinite. $X \cup Y$ is cofinite $\Rightarrow \overline{X \cup Y}$ is finite $\Rightarrow \overline{X} \cap \overline{Y}$ is finite. Since X and Y are NOT cofinite, then $\overline{X}$ and $\overline{Y}$ are NOT finite. It's not true that A, B NOT finite $\Rightarrow A \cap B$ finite. For example, take $A = \{n \in \mathbb{N} | n > 2\}$ and $B = \{n \in \mathbb{N} | n > 3\}$; then $A \cap B = B$ is NOT finite.

Question 3. In what follows, $A \subseteq \mathbb{N}$.

1. Prove that if there exists a bijection $f \in (\mathbb{N} \to A)$, then A is infinite.

2. Can you provide an example of an infinite set A and of a function $f \in (\mathbb{N} \to A)$ which is neither injective nor surjective?

Answer 3.1.

1. By definition, two sets have the same cardinality iff there exists a bijection that puts in correspondence the elements of the first set with the elements of the second. Suppose that there exists a bijection $f \in (\mathbb{N} \to A)$ and suppose by contradiction that A is finite (i.e. $\exists m \in \mathbb{N}. |A| = m$). It follows that also $\mathbb{N}$ is finite. We have reached a contradiction, therefore A is infinite.

2. Take A as the set of even numbers and take

$$f(n) = \begin{cases} 0 \ \textit{if } n \textit{ is even} \\ 2 \ \textit{if } n \textit{ is odd} \end{cases}$$

The function f is neither injective ($f(1) = f(3)$) nor surjective ($4 \in A$ has no preimage)