Computability Assignment
Year 2013/14 - Number 6

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1 Question

Remember that for all $A \subseteq \mathbb{N}$, $\overline{A} = \mathbb{N} \setminus A$, and id_A is the identity function on A.

Let $f \in (\mathbb{N} \to \mathbb{N})$ and let $A = \{f(n) | n \text{ is a prime number}\}$.

1. Characterize the elements of the set $\overline{A}$ (i.e. find a property p such that $\overline{A} = \{n | p(n)\}$). Notice that p could be a conjunction of many "simpler" properties.
2. Define a function $g \in (A \to \mathbb{N})$ such that $f \circ g = \mathsf{id}_A$.

1.1 Answer

1. $A = \{n | n = ab, n \in \mathbb{N}, a \neq 1 \wedge b \neq 1\}$
2. Let $f_{pedice} : A \rightsquigarrow \mathbb{N}$ such that for all $a \in A.f_{pedice}(a) = f(a)$. Then $g(x) = f^{-1}_{pedice}(x).\forall x \in A$

2 Question

Let $A = \{n | \exists m \in \mathbb{N}.\ n = m^2\}$ and $B = \{2n | n \in \mathbb{N}\}$. Following the steps outlined below, define a bijection $f \in (\mathbb{N} \to \mathbb{N})$ such that $f(A) = B$ and $f(\overline{A}) = \overline{B}$.

1. Provide a bijection $g \in (A \to \mathbb{N})$.
2. Provide a bijection $h \in (\mathbb{N} \to B)$.
3. Argue that there exists a bijection $g' \in (\overline{A} \to \mathbb{N})$.

4. Provide a bijection $h' \in (\mathbb{N} \to \overline{B})$.

5. Prove that the function $f \in (\mathbb{N} \to \mathbb{N})$ defined as

$$f(n) = \begin{cases} (h \circ g)(n) & if\ n \in A \\ (h' \circ g')(n) & if\ n \in \overline{A} \end{cases}$$

satisfies all the desired properties.

2.1 Answer

1. We take $g(n) = m$, where m is the number such that $m^2 = n$ that appears in the definition of A.

2. We take $h(n) = 2n$.

3. The set $\overline{A}$ is the set of n such that $n \in \mathbb{N}, \sqrt{n} \notin \mathbb{N}$. A bijection could be $g'(n) = min(\overline{A} \setminus \cup_{i=0}^{n-2} f(i))$, with $g'(0) = min(\overline{A})$. In other words, the set $\overline{A}$ is an ordered set, and we have a bijection between the i-th member of the set and the natural number i.

4. We take $h'(n) = 2n + 1$.

5. If $n \in A$, then $\exists m \in \mathbb{N}$ such that $n = m^2$. By definition, $g(n) = m$, and $h(g(n)) = h(m) = 2m$. Note that $h \circ g$ is a bijection by construction, as it is the composition of two bijections. Dually, if $n \notin A$, and is a natural number, then by definition $n \in \overline{A}$ (thus f is indeed total), and $i_n = g'(n)$ is the n-th smallest element of $\overline{A}$, whereas $h'(g'(n)) = 2i_n + 1$. In this case, too, we have that the function is a bijection, because it is the compostion of two bijections. The function is total, since the element can either belong to A or to its complement. Furthermore, A and $\overline{A}$ is a partition of $\mathbb{N}$, and B and $\overline{B}$ as well, thus $f : \mathbb{N} \to \mathbb{N}$. And we are indeed sure, by construction, that $f(A) = B$ and $f(\overline{A}) = \overline{B}$, since if $a \in A, f(a) = h \circ g(a)$, and $h \circ g : A \to B$, whereas if $a \notin A$, then $a \in \mathbb{N} \setminus A, f(a) = h' \circ g'(a)$, and $h' \circ g' : \mathbb{N} \setminus A \to \mathbb{N} \setminus B$.