Computability Assignment
Year 2013/14 - Number 6

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1 Question

Remember that for all $A \subseteq \mathbb{N}$, $\overline{A} = \mathbb{N} \setminus A$, and id_A is the identity function on A.

Let $f \in (\mathbb{N} \to \mathbb{N})$ and let $A = \{f(n)|n \text{ is a prime number}\}$.

1. Characterize the elements of the set $\overline{A}$ (i.e. find a property p such that $\overline{A} = \{n|p(n)\}$). Notice that p could be a conjunction of many "simpler" properties.
2. Define a function $g \in (A \to \mathbb{N})$ such that $f \circ g = \mathsf{id}_A$.

1.1 Answer

1. Since f is a total function from $\mathbb{N}$ to $\mathbb{N}$ itself, f must be defined also for all those $n \in \mathbb{N}$ such that n is not a prime number — i.e. n is a composite number, or more formally $n \in \mathbb{N} \mid \exists a, b \in \mathbb{N}.ab = n$. Having no more information about f , we have to take into account that $f(n)$, for those n's composite, can go to both the sets A and $\bar{A}$. But since A contains all the $f(n)$'s with n prime, $\bar{A}$ cannot contains other $f(x)$'s with n prime, so it must contains all the $f(n)$'s with n composite. Finally we can say that p is just this property, — i.e. $p =$ "n is a composite number" — and so $\bar{A} = \{f(n) \mid p(n)\}$.
2. $g(n) = \begin{cases} f^{-1}(n) & \text{if n is a prime number} \\ undefined & \text{otherwise} \end{cases}$

2 Question

Let $A = \{n | \exists m \in \mathbb{N}.\ n = m^2\}$ and $B = \{2n | n \in \mathbb{N}\}$. Following the steps outlined below, define a bijection $f \in (\mathbb{N} \to \mathbb{N})$ such that $f(A) = B$ and $f(\overline{A}) = \overline{B}$.

1. Provide a bijection $g \in (A \to \mathbb{N})$.
2. Provide a bijection $h \in (\mathbb{N} \to B)$.
3. Argue that there exists a bijection $g' \in (\overline{A} \to \mathbb{N})$.
4. Provide a bijection $h' \in (\mathbb{N} \to \overline{B})$.
5. Prove that the function $f \in (\mathbb{N} \to \mathbb{N})$ defined as

$$f(n) = \begin{cases} (h \circ g)(n) & if\ n \in A \\ (h' \circ g')(n) & if\ n \in \overline{A} \end{cases}$$

satisfies all the desired properties.

2.1 Answer

1. $g(n) = \sqrt{n}$, which is bijective (surjectivity holds within the constraints of having A as domain).
2. $h(n) = 2n$, which is bijective.
3. It's possible to find a such g'because, thanks to the enumeration method, we can assert that it's always possible to find a bijection between $\mathbb{N}$ and any its subset.
4. $h'(n) = 2n + 1$, which is bijective.
5. We can simply reason as follows:
   (a) $n \in A \implies f(n) = h(g(n)) \implies f(A) = h(g(A)) = h(\mathbb{N}) = B$;
   (b) $n \notin A \implies f(n) = h'(g'(n)) \implies f(\bar{A}) = h'(g'(\bar{A})) = h'(\mathbb{N}) = \bar{B} = \mathbb{N} \setminus B$.