Computability Assignment
Year 2013/14 - Number 6

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1 Question

Remember that for all $A \subseteq \mathbb{N}$, $\overline{A} = \mathbb{N} \setminus A$, and id_A is the identity function on A.

Let $f \in (\mathbb{N} \to \mathbb{N})$ and let $A = \{f(n)|n \text{ is a prime number}\}$.

1. Characterize the elements of the set $\overline{A}$ (i.e. find a property p such that $\overline{A} = \{n|p(n)\}$). Notice that p could be a conjunction of many "simpler" properties.
2. Define a function $g \in (A \to \mathbb{N})$ such that $f \circ g = \mathsf{id}_A$.

1.1 Answer

1. Let's denote $P := \{p \in \mathbb{N}|p \textit{ is prime}\}$ in such a way that $f(P) = A$. $\overline{A} = \{n \in \mathbb{N}|n \notin A\} = \{f(n) \in \mathbb{N}|n \textit{ is not a prime number}\} = \{f(0)\} \cup \{f(1)\} \cup \{f(n) \in \mathbb{N}|\exists p, q \in P \textit{ such that } f(n) = p \cdot q\}$.
2. We want that $\forall n \in A.n = f(g(n))$. But since $n \in A$, by definition $\exists p \in P.n = f(p)$. The first condition becomes $n = f(g(n)) = f(g(f(p)))$. Hence $g(f(p))$ has to be equal to p. This is easily done by taking $g \in (A \to \mathbb{N})$ such that $g \circ (f\ |_P) = \mathsf{id}_P$.

2 Question

Let $A = \{n|\exists m \in \mathbb{N}.\ n = m^2\}$ and $B = \{2n|n \in \mathbb{N}\}$. Following the steps outlined below, define a bijection $f \in (\mathbb{N} \to \mathbb{N})$ such that $f(A) = B$ and $f(\overline{A}) = \overline{B}$.

1. Provide a bijection $g \in (A \to \mathbb{N})$.
2. Provide a bijection $h \in (\mathbb{N} \to B)$.
3. Argue that there exists a bijection $g' \in (\overline{A} \to \mathbb{N})$.
4. Provide a bijection $h' \in (\mathbb{N} \to \overline{B})$.
5. Prove that the function $f \in (\mathbb{N} \to \mathbb{N})$ defined as

$$f(n) = \begin{cases} (h \circ g)(n) & if\ n \in A \\ (h' \circ g')(n) & if\ n \in \overline{A} \end{cases}$$

   satisfies all the desired properties.

2.1 Answer

1. $\forall n \in A$ let's define $g(n) = g(m^2) = m$.
2. $\forall n \in \mathbb{N}$let's define $h(n) = 2n$.
3. Very intuitivly we can write $A = \{0, 1, 4, 9, 16, 25, ...\}$ then $\overline{A} = \{2, 3, 5, 6, 7, 8, 10, ...\}$. Hence we can define $g'(2) = 0,\ g'(3) = 1,\ g'(5) = 2,\ g'(6) = 3$ and so on.
   This is a bijection.
4. $B = \{2n | n \in \mathbb{N}\}$ then $\overline{B} = \{2n + 1 | n \in \mathbb{N}\}$. Hence $\forall n \in \mathbb{N}$ we define $h'(n) = 2n + 1$.
5. We have to do two proofs:
   (a) $n \in A$: $f(n) = h(g(n)) = h(g(m^2)) = h(m) = 2m \in B$.
   (b) $n \in \overline{A} : f(n) = h'(g'(n)) = h'(m) = 2m + 1 \in \overline{B}$ for some $m \in \mathbb{N}$