Computability Assignment
Year 2013/14 - Number 6

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1 Question

Remember that for all $A \subseteq \mathbb{N}$, $\overline{A} = \mathbb{N} \setminus A$, and id_A is the identity function on A.

Let $f \in (\mathbb{N} \to \mathbb{N})$ and let $A = \{f(n)|n \text{ is a prime number}\}$.

1. Characterize the elements of the set $\overline{A}$ (i.e. find a property p such that $\overline{A} = \{n|p(n)\}$). Notice that p could be a conjunction of many "simpler" properties.
2. Define a function $g \in (A \to \mathbb{N})$ such that $f \circ g = \mathsf{id}_A$.

1.1 Answer

1. $\overline{A} = \{n \in \mathbb{N}|\forall m \in \mathbb{N}.\ (f(m) = n \Rightarrow m \text{ not prime})\}$
2. $g(a) = \min(\{m \in \mathbb{N}|f(m) = a\})$ Note g is properly defined for each element of A (because each element of A has a counter-image w.r.t. f by definition of A)

2 Question

Let $A = \{n|\exists m \in \mathbb{N}.\ n = m^2\}$ and $B = \{2n|n \in \mathbb{N}\}$. Following the steps outlined below, define a bijection $f \in (\mathbb{N} \to \mathbb{N})$ such that $f(A) = B$ and $f(\overline{A}) = \overline{B}$.

1. Provide a bijection $g \in (A \to \mathbb{N})$.
2. Provide a bijection $h \in (\mathbb{N} \to B)$.

3. Argue that there exists a bijection $g' \in (\overline{A} \to \mathbb{N})$.
4. Provide a bijection $h' \in (\mathbb{N} \to \overline{B})$.
5. Prove that the function $f \in (\mathbb{N} \to \mathbb{N})$ defined as

$$f(n) = \begin{cases} (h \circ g)(n) & if\ n \in A \\ (h' \circ g')(n) & if\ n \in \overline{A} \end{cases}$$

satisfies all the desired properties.

2.1 Answer

1. $g(a) = \sqrt{a}$ (note that, by definition of A, $\forall a \in A.\ \sqrt{a} \in \mathbb{N}$)
2. $h(a) = 2 \cdot a$
3. It is always possible to find a bijection between $\mathbb{N}$ and any non-finite $D \subseteq \mathbb{N}$. This can be done by "enumerating", i.e. defying an order over the elements of the subset (the simplest being plain "less than"): the simply associate each element in D to its 0-based index in the enumeration, and symmetrically associate each natural i to the $(i+1)^{th}$ element in the sequence.
4. $h'(n) = 2 \cdot n + 1$
5. Consider the two cases for which f is defined:

   (a) if $n \in A$, then $f(n) = (h \circ g)(n) = h(g(n)) = 2 \cdot \sqrt{n}$. Thus $f(A) = h(g(A)) = h(\mathbb{N}) = B$. Note this is a bijection (composition of two bijections);

   (b) if $n \in \overline{A}$, then $f(n) = (h' \circ g')(n) = h'(g'(n)) = 2 \cdot g'(n) + 1$. Note that, by construction, $g'(\overline{A}) = \mathbb{N}$, so $f(\overline{A}) = h'(g'(\overline{A})) = h'(\mathbb{N}) = \overline{B}$; note this is a bijecition as well.

Note f is the union of 2 bijections between disjoint sets, so it is itself a bijection.