Year 2013/14 - Number 6

October 25, 2013

Please keep this file anonymous: do not write your name inside this file.
More information about assignments at
`http://disi.unitn.it/zunino/teaching/computability/assignments`.

Question 1. Remember that for all $A \in \mathbb{N}$, $\bar{A} = \mathbb{N} \setminus A$, and id_A is the identity function on A. Let $f \in (\mathbb{N} \to \mathbb{N})$ and let $A = \{f(n)|n \text{ is a prime number}\}$.

1. Characterize the elements of the set $\bar{A}$ (i.e. find a property p such that $\bar{A} = \{n|p(n)\}$). Notice that p could be a conjunction of many "simpler" proprieties.
2. Define a function $g \in (A \to \mathbb{N})$ such that $f \circ g = id_A$.

Answer 1.1.

1. We define the property p as $p(n) \equiv (n = f(m) \text{ for some } n \in \mathbb{N}) \wedge (m = pq \text{ such that } p, q \neq 1 \wedge p, q \in \mathbb{N})$.
2. Since f is a total function and thanks to the definition of A, we can state that the restriction $f_{|B} \in (B \to A)$, where B is the set of prime numbers, is bijective. We define g as $g(x) = f_{|B}^{-1}(x)$.

Question 2. Let $A = \{n|\exists m \in \mathbb{N}.n = m^2\}$ and $B = \{2n|n \in \mathbb{N}\}$. Following the steps outlined below, define a bijection $f \in (\mathbb{N} \to \mathbb{N})$ such that $f(A) = B$ and $f(\bar{A}) = \bar{B}$.

1. Provide a bijection $g \in (A \to \mathbb{N})$.
2. Provide a bijection $h \in (\mathbb{N} \to B)$.
3. Argue that there exists a bijection $g' \in (\bar{A} \to \mathbb{N})$.

4. Provide a bijection $h' \in (\mathbb{N} \to \bar{B})$.

5. Prove that the function $f \in (\mathbb{N} \to \mathbb{N})$ defined as

$$f(n) = \begin{cases} (h \circ g)(n) \ \textit{if}\ n \in A \\ (h' \circ g')(n) \ \textit{if}\ n \in \bar{A} \end{cases}$$

satisfies all the desired proprieties.

Answer 2.1.

1. We define $g \in (A \to \mathbb{N})$ as $g(n) = \sqrt{n}$. Notice that $g(n) \in \mathbb{N}$ because $n = m^2, m \in \mathbb{N} \Rightarrow g(n) = m \in \mathbb{N}$.
   We prove the surjectivity of g: $\forall x \in \mathbb{N}$, $\exists y = x^2 \in A$ such that $g(y) = x$, therefore g is surjective.
   We prove the injectivity of g: let $x, x' \in A$ such that $x \neq x'$. Then $g(x) = \sqrt{x} \neq \sqrt{x'} = g(x')$, therefore g is injective.

2. We define $h \in (\mathbb{N} \to B)$ as $h(n) = 2n$. Clearly, the function is bijective.

3. Notice that $\bar{A}$ is the set of natural number n, for which $\sqrt{n} \notin \mathbb{N}$. We could define the function g' in this way:

$$\begin{gathered} \bar{A} \xrightarrow{g} \mathbb{N} \\ 2 \to 1 \\ 3 \to 2 \\ 5 \to 3 \\ 6 \to 4 \end{gathered}$$

   and so on.. That function g will be bijective.

4. We define $h' \in (\mathbb{N} \to \bar{B})$ as $h'(n) = 2n + 1$. Clearly, the function is bijective.

5. We prove that the function f defined above, satisfies all the desired properties.
   First of all notice that $dom(f) = A \cup \bar{A} = \mathbb{N}$ and $ran(f) = (\{h(g(x)) | x \in A\} \cup \{h'(g'(x)) | x \in \bar{A}\}) \subseteq B \cup \bar{B} = \mathbb{N}$, therefore $f \in (\mathbb{N} \to \mathbb{N})$. Then observe that f is bijective because composition of bijective functions.
   We state that $f(A) = B$: $\forall x \in A$, $f(x) = h(g(x)) = h(\sqrt{x}) = 2\sqrt{x} \in B$ and $\forall x \in B\ \exists y = (\frac{x}{2})^2 \in A$ such that $f(y) = x$.
   We state that $f(\bar{A}) = \bar{B}$: $\forall x \in \bar{A}$, $f(x) = h(g(x)) \in \bar{B}$ and $\forall x \in \bar{B}$ $\exists y = g'^{-1}(\frac{x-1}{2})$ such that $f(y) = x$ (notice that $\frac{x-1}{2} \in \mathbb{N}$ and g' bijective, so that y actually exists).

document