Computability Assignment
Year 2012/13 - Number 5

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1 Question

Prove by induction that $\forall k \in \mathbb{N}.\ 9^k - 2^k$ is a multiple of 7. Follow the steps outlined below.

1. Prove that, if $k = 0$, then $9^0 - 2^0$ is a multiple of 7. This is the basis of the induction.

2. Now, suppose that for a *generic* natural number n, it is true that $9^n - 2^n$ is a multiple of 7. By only using this *inductive hypothesis*, prove that $9^{n+1} - 2^{n+1}$ is a multiple of 7. To do so, use the identity:

   $$9^{n+1} - 2^{n+1} = 9^{n+1} - 9^n \cdot 2 + 9^n \cdot 2 - 2^{n+1}$$

   and a clever factorization of the right-hand side of the equality. Remember that, at some point, you are expected to use the *inductive hypothesis*.

1.1 Answer

1. If $k = 0$ then $9^0 - 2^0 = 0$ that is a multiple of 7 in fact $0 = 7 \cdot 0$;

2. Inductive Step: suppose that the propriety holds for k, we prove that it holds for $k+1$ too. Notice that $9^{k+1} - 2^{k+1} = 9^{n+1} - 9^n \cdot 2 + 9^n \cdot 2 - 2^{n+1} = 9^n(9-2) + 2(9^n - 2^n) = 7 \cdot 9^n + 2 \cdot 7c$ by the inductive hypothesis the propriety holds for k, therefore $(9^n - 2^n) = 7c$ for some $c \in \mathbb{N}$. It follows that $9^{k+1} - 2^{k+1} = 7(9^n + 2c)$, that is a multiple of 7.

2 Preliminaries

Let $P(k)$ be the property "$\forall n, m \in \mathbb{N}.\ max(n,m) = k$ implies $n = m$". The following is a proof by induction that $\forall k \in \mathbb{N}.P(k)$.

1. Basis of the induction: if $max(n,m) = 0$ then $n = m = 0$, as we wanted.

2. Inductive step: suppose that $P(k)$ is true for a generic natural number k; we want to prove that this implies $P(k+1)$, i.e. that for all natural numbers n, m such that $max(n,m) = k+1$, $n = m$. So, let $n, m \in \mathbb{N}$ satisfy $max(n,m) = k+1$. Then $max(n-1, m-1) = max(n,m) - 1 = k$. By the induction hypothesis, it follows that $n - 1 = m - 1$, and therefore $n = m$. This proves $P(k+1)$, so the induction step is complete.

3 Question

Is the above proof correct? If not, can you tell what is wrong with it?

3.1 Answer

The proof is not correct because "$\forall n, m \in \mathbb{N}.\ max(n,m) = k \implies n = m$" isn't valid $\forall n, m \in \mathbb{N}$. For example $max(1,1) = 1$ but also $max(1,0) = 1$ and $max(0,1) = 1$. If we assume that $max = 2$ this may result from couples $(0,2)$, $(2,0)$, $(1,2)$, $(2,1)$ and $(2,2)$.