Computability Assignment
Year 2012/13 - Number 5

Please keep this file anonymous: do not write your name inside this file.

More information about assignments at http://disi.unitn.it/∼zunino/teaching/computability/assignments

Please do not submit a file containing only the answers; edit this file, instead, filling the answer sections.

1 Question

Prove by induction that $\forall k \in \mathbb{N}.\ 9^k - 2^k$ is a multiple of 7. Follow the steps outlined below.

1. Prove that, if $k = 0$, then $9^0 - 2^0$ is a multiple of 7. This is the basis of the induction.
2. Now, suppose that for a *generic* natural number n, it is true that $9^n - 2^n$ is a multiple of 7. By only using this *inductive hypothesis*, prove that $9^{n+1} - 2^{n+1}$ is a multiple of 7. To do so, use the identity:

   $$9^{n+1} - 2^{n+1} = 9^{n+1} - 9^n \cdot 2 + 9^n \cdot 2 - 2^{n+1}$$

   and a clever factorization of the right-hand side of the equality. Remember that, at some point, you are expected to use the *inductive hypothesis.*

1.1 Answer

The base case is $k = 0, 9^0 - 2^0 = 0 = 0 \cdot 7$, verified.
The inductive step, for a generic n, is:
$9^{n+1} - 2^{n+1} = 9^{n+1} - 9^n \cdot 2 + 9^n \cdot 2 - 2^{n+1} = 9 \cdot 9^n - 9^n \cdot 2 + 9^n \cdot 2 - 2 \cdot 2^n = 7 \cdot 9^n + 2 \cdot (9^n - 2^n) = 7 \cdot 9^n + 2 \cdot c \cdot 7 = 7 \cdot (9^n + 2 \cdot c) = 7 \cdot d$
q.e.d.

2 Preliminaries

Let $P(k)$ be the property "$\forall n, m \in \mathbb{N}.\ max(n, m) = k$ implies $n = m$". The following is a proof by induction that $\forall k \in \mathbb{N}.P(k)$.

1. Basis of the induction: if $max(n, m) = 0$ then $n = m = 0$, as we wanted.

2. Inductive step: suppose that $P(k)$ is true for a generic natural number k; we want to prove that this implies $P(k + 1)$, i.e. that for all natural numbers n, m such that $max(n, m) = k + 1$, $n = m$. So, let $n, m \in \mathbb{N}$ satisfy $max(n, m) = k+1$. Then $max(n-1, m-1) = max(n, m)-1 = k$. By the induction hypothesis, it follows that $n - 1 = m - 1$, and therefore $n = m$. This proves $P(k + 1)$, so the induction step is complete.

3 Question

Is the above proof correct? If not, can you tell what is wrong with it?

3.1 Answer

The proof is wrong. The problem is in the "By the induction hypothesis, it follows that $n-1 = m-1$, and therefore $n = m$" part. If we consider $k+1 = 1$, thus $k = 0$, and we consider $n = 0, m = 1$ we have that $n - 1 = -1, m - 1 = 0$, thus we *cannot* use the hypothesis, since it requires that $n \in \mathbb{N}, m \in \mathbb{N}$, not $\mathbb{Z}$.