Year 2013/14 - Number 5

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Question 1. Prove by induction that $\forall k \in \mathbb{N}.9^k - 2^k$ is a multiple of 7. Follow the steps outlined below.

1. Prove that, if $k = 0$, then $9^0 - 2^0$ is a multiple of 7. This is the basis of the induction.

2. Now, suppose that for a *generic* natural numebr n, it is true that $9^n - 2^n$ is a multiple of 7.By only using this *inductive hypothesis*, prove that $9^{n+1} - 2^{n+1}$ is a multiple of 7. To do so, use the identity:

$$9^{n+1} - 2^{n+1} = 9^{n+1} - 9^n \cdot 2 + 9^n \cdot 2 - 2^{n+1}$$

and a clever factorization of the right-hand side of the equality. Rember that, at some point, you are expected to use the *inductive hypothesis*.

Answer 1.1.

1. Base Case: $9^0 - 2^0 = 7$, that is a multiple of 7.

2. Inductive Step: suppose that the propriety holds for k, we prove that it holds for $k+1$ too. Notice that $9^{k+1} - 2^{k+1} = 9^{n+1} - 9^n \cdot 2 + 9^n \cdot 2 - 2^{n+1} = 9^n(9 - 2) + 2(9^n - 2^n) = 7 \cdot 9^n + 2 \cdot 7c$ (by the inductive hypothesis the propriety holds for k, therefore $(9^n - 2^n) = 7c$ for some $c \in \mathbb{N}$). It follows that $9^{k+1} - 2^{k+1} = 7(9^n + 2c)$, that is a multiple of 7.

Preliminaries 2. Let $P(k)$ be the propriety "$\forall n, m \in \mathbb{N}.max(n, m) = k$ *implies* $n = m$". The following is a proof by induction that $\forall k \in \mathbb{N}.P(k)$.

1. Basis of the induction: if $max(n, m) = 0$ then $n = m = 0$, as we wanted.

2. Inductive step: suppose that $P(k)$ is true for a generic natural number k; we want to prove that this implies $P(k+1)$, i.e. that for all natural numbers n, m such that $max(n, m) = k + 1$, $n = m$. So let $n, m \in \mathbb{N}$ satisfy $max(n, m) = k + 1$. Then $max(n - 1, m - 1) = max(n, m) - 1 = k$. By the induction hypothesis, it follows that $n - 1 = m - 1$, and therefore $n = m$. This proves $P(k + 1)$, so the induction step is complete.

Question 3. Is the above proof correct? If not, can you tell what is wrong with it?

Answer 3.1. The proof isn't correct due to the implication "$max(n, m) = k + 1 \Rightarrow max(n - 1, m - 1) = max(n, m) - 1 = k$". In fact, n, m are generic natural numbers, but not all of them have a predecessor. In particular, when $n = m = 0$, $n - 1$ and $m - 1$ aren't natural numbers. This means that the implication isn't always true, therefore it's not valid. The inductive step is wrong.