Computability Assignment
Year 2012/13 - Number 5

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1 Question

Prove by induction that $\forall k \in \mathbb{N}.\ 9^k - 2^k$ is a multiple of 7. Follow the steps outlined below.

1. Prove that, if $k = 0$, then $9^0 - 2^0$ is a multiple of 7. This is the basis of the induction.
2. Now, suppose that for a *generic* natural number n, it is true that $9^n - 2^n$ is a multiple of 7. By only using this *inductive hypothesis*, prove that $9^{n+1} - 2^{n+1}$ is a multiple of 7. To do so, use the identity:

   $$9^{n+1} - 2^{n+1} = 9^{n+1} - 9^n \cdot 2 + 9^n \cdot 2 - 2^{n+1}$$

   and a clever factorization of the right-hand side of the equality. Remember that, at some point, you are expected to use the *inductive hypothesis.*

1.1 Answer

1. if $k = 0$, then $9^0 - 2^0 = 0$ is a multiple of 7 (in fact, $0 = 7 \times 0$)
2. Supposing $\exists c \in \mathbb{Z}.\ 9^n - 2^n = 7 \cdot c$ for a generic $n \in \mathbb{N}$,

   $$9^{n+1}-2^{n+1} = 9^{n+1}-9^n \cdot 2+9^n \cdot 2-2^{n+1} = 9^n (9-2)+2(9^n - 2^n) = 7 \cdot 9^n + 2(7 \cdot c) = 7(9^n + 2c)$$

   (where c is the integer which exist by inductive hypothesis). We thus proved $\exists d \in \mathbb{Z}.\ 9^{n+1} - 2^{n+1} = 7 \cdot d$. The thesis is thus proved by induction.

2 Preliminaries

Let $P(k)$ be the property "$\forall n, m \in \mathbb{N}.\ max(n, m) = k$ implies $n = m$". The following is a proof by induction that $\forall k \in \mathbb{N}.P(k)$.

1. Basis of the induction: if $max(n, m) = 0$ then $n = m = 0$, as we wanted.

2. Inductive step: suppose that $P(k)$ is true for a generic natural number k; we want to prove that this implies $P(k + 1)$, i.e. that for all natural numbers n, m such that $max(n, m) = k + 1$, $n = m$. So, let $n, m \in \mathbb{N}$ satisfy $max(n, m) = k+1$. Then $max(n-1, m-1) = max(n, m)-1 = k$. By the induction hypothesis, it follows that $n - 1 = m - 1$, and therefore $n = m$. This proves $P(k + 1)$, so the induction step is complete.

3 Question

Is the above proof correct? If not, can you tell what is wrong with it?

3.1 Answer

The theorem is obviously wrong. The problem is in the inductive step: let's follow the proof at the step when $k + 1 = 1$, so$k = 0$

- "let $n, m \in \mathbb{N}$ satisfy $max(n, m) = k + 1$": consider $n = 1,\ m = 0$ (which satisfy the condition)

- "Then $max(n - 1, m - 1) = \ldots = k$": in our case, $max(n - 1, m - 1) = max(0, -1) = 0 = k$

- "By the induction hypothesis, it follows that $n-1 = m-1$" (this is wrong): in the induction step, we can assume that $\forall n, m \in \mathbb{N}.\ (max(n, m) = k \to n = m)$, but we can NOT assume $\forall n, m \in \mathbb{Z}.\ (max(n, m) = k \to n = m)$: in our case, $n - 1 = -1 \notin \mathbb{N}$, so the induction hypothesis does not apply.

- Starting from this wrong assumption, the false thesis is then proved.