Computability Assignment
Year 2012/13 - Number 4

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1 Question

Let A, B be sets and suppose that $A \leftrightarrow B$ (i.e. there exists a bijection $f \in (A \to B)$). Show that for all sets C, $(C \to (A \times A)) \leftrightarrow (C \to (A \times B))$.

1.1 Answer

By hypothesis $\exists f \in (A \leftrightarrow B)$. I define $g \in ((A \times A) \to (A \times B))$ such that $g(a_1, a_2) = (a_1, f(a_2))$. Of course g is a bijection, since its component are bijections (identity and f).

Now I define $H \in ((C \to (A \times A)) \to (C \to (A \times B)))$ such that $H(q) := g \circ q$. $q \in (C \to (A \times A))$, so that it makes sense to do $H(q)$. Since $g \in ((A \times A) \to (A \times B))$ and $q \in (C \to (A \times A))$ then $g \circ q \in (C \to (A \times B))$. Hence H is well defined.

Now I proove that H is a bijection:

1. H injective: Let $q_1, q_2 \in (C \to (A \times A))$ such that $H(q_1) = H(q_2)$ and $q_1(c) = (a_{11}, a_{12}), q_2 = (a_{21}, a_{22})$. I want to proove that $q_1 = q_2$.
   (a) $H(q_1)(c) = g \circ q_1(c) = g(q_1(c)) = g((a_{11}, a_{12})) = (a_{11}, f(a_{12}))$;
   (b) $H(q_2)(c) = g \circ q_2(c) = g(q_2(c)) = g((a_{21}, a_{22})) = (a_{21}, f(a_{22}))$;
   (c) By setting $H(q_1) = H(q_2)$, we obtain directly that $a_{11} = a_{21}$ and $f(a_{12}) = f(a_{22})$. Hence, by the fact that f is injective we are done.
2. H surjective: Let $p \in (C \to (A \times B))$ such that $p(c) = (a, b)$ where $b \in B$. Since f is surjective we know that $\exists a_b \in A$ such that $b = f(a_b)$. Then, if I take $q(c) = (a, a_b) \in (C \to (A \times A))$ we have that $H(q)(c) = g \circ q(c) = g(q(c)) = g((a, a_b)) = (a, b) = p(c)$.

2 Question

1. Doeas a surjective function $f \in (\mathbb{N} \rightarrow (\mathbb{N} \rightarrow \{0,1,2,3\}))$ exist?
2. Does an injective function $f \in (\mathcal{P}(\mathbb{N}) \rightsquigarrow \mathbb{N})$ exist?
3. Does an injective function $f \in (\mathcal{P}(\mathbb{N}) \rightarrow \mathbb{N})$ exist?

Justify your answers.

2.1 Answer

1. No: Let $f \in (\mathbb{N} \rightarrow (\mathbb{N} \rightarrow \{0,1,2,3\}))$. Let's define $g \in (\mathbb{N} \rightarrow \{0,1,2,3,4\})$ such that $g(n) := \begin{cases} 0 & if \quad f(n)(n) = 4 \\ f(n)(n)+1 & if \quad f(n)(n) \neq 4 \end{cases}$. I want to proove that f isn't surjective, i.e. $\forall n \in \mathbb{N}. g \neq f(n)$. By contraddiction let $g = f(n)$ for some n. Then $g(n) = f(n)(n)$. We have now two cases:
   (a) $f(n)(n) = 4 \Rightarrow g(n) = 0$. Since $g(n) = f(n)(n)$ I obtain $4 = 0$, which is absurd;
   (b) $f(n)(n) \neq 4 \Rightarrow g(n) = f(n)(n) + 1$. Since $g(n) = f(n)(n)$ I obtain $f(n)(n) = f(n)(n) + 1$, which is absurd too.
2. Yes: $f(\{n\}) = n$ is the requested function because $f \in (\mathcal{P}(\mathbb{N}) \rightsquigarrow \mathbb{N})$ and it's injective. Note that it is a partial function because it takes as input only the subsets of $\mathbb{N}$ of the form $\{n\}$ and because $f(\emptyset)$ is undefinied.
3. No: By contraddiction f is injective. Then $f|_F$ is bijective where $F = ran(f) \subseteq \mathbb{N}$. Then $F \leftrightarrow \mathcal{P}(\mathbb{N})$. But F is an infinite subset of $\mathbb{N}$, then $F \leftrightarrow \mathbb{N}$. Absurd.

3 Question

Let A, B be nonempty sets and let $f \in (A \rightarrow B)$. Define a function $g \in (B \rightsquigarrow A)$ such that $\text{dom}(g) \neq \emptyset$ and for all $b \in \text{dom(g)}$, $(f \circ g)(b) = b$.

3.1 Answer

The unique constraints that has to be putted on g is that $dom(g) = f(A)$. Of course $dom(g) \neq \emptyset$ and $\forall b \in dom(g)\ g(b) \in f^{-1}(b)$, so that $f \circ g(b) = f(g(b)) = b$.