Year 2013/14 - Number 4

October 11, 2013

Please keep this file anonymous: do not write your name inside this file.
More information about assignments at
`http://disi.unitn.it/zunino/teaching/computability/assignments`.

Question 1. Let A, B be sets and suppose that $A \leftrightarrow B$ (i.e. there exists a bijection $f \in (A \to B)$). Show that for all sets C, $(C \to (A \times A)) \leftrightarrow (C \to (A \times B))$.

Answer 1.1. Let $Left \in (X \times Y \to X)$ and $Right \in (X \times Y \to Y)$ be the projection maps from a pair onto the left and right component respectively:
$Left(< x, y >) = x$
$Right(< x, y >) = y$
Let $h \in (A \leftrightarrow B)$ be a bijection between the given sets A and B. Consider $f \in (C \to (A \times A))$. It follows that $\forall x \in C$, $f(x) =< a, b >$, where $a \in A, b \in A$. We want to build a bijection Φ, such that $\forall f \in (C \to (A \times A))$, $\Phi(f) \in (C \to (A \times B))$. We define $\Phi(f)(x) =< Left(f(x)), h(Right(f(x))) >$, for all $x \in C$.
Notice that $h(Right(f(x)) \in B$ because of the definition of h, therefore the function $\Phi(f)$ is actually an element of $(C \to (A \times B))$. We can state that Φ is injective because h is injective, therefore different function f, f' are mapped in different g, g'. We can also state that Φ is surjective because h is bijective, therefore it exists an inverse function h^{-1} which maps every element $b \in B$ onto an element $a \in A$:
Consider $g \in (C \to (A \times B))$, we define $\Phi^{-1}(g)(x) =< Left(g(x)), h^{-1}(Right(g(x))) >$.

Question 2.

1. Does a surjective function $f \in (\mathbb{N} \to (\mathbb{N} \to \{0, 1, 2, 3\}))$ exist?
2. Does an injective function $f \in (P(\mathbb{N}) \rightsquigarrow \mathbb{N})$ exist?

3. Does an injective function $f \in (P(\mathbb{N}) \to \mathbb{N})$ eist?

Justify your answers.

Answer 2.1.

1. no. Notice that it exists a bijection $h \in (P(\mathbb{N}) \leftrightarrow (\mathbb{N} \to \{0,1\})$ defined $\forall C \in P(\mathbb{N})$ $h(C)(x) = 1$ if $x \in C$, 0 otherwise.Notice also that $(\mathbb{N} \to \{0,1\}) \subset (\mathbb{N} \to \{0,1,2,3\}))$. The diagonalization argument used in the proof of Cantor's theorem, states that there is no surjective function between A and $P(A)$, therefore, in particular there is no surjective function between $\mathbb{N}$ and $P(\mathbb{N})$. It follows that there cannot be a surjective function $f \in (\mathbb{N} \to (\mathbb{N} \to \{0,1\}))$, therefore between $\mathbb{N}$ and $\{0,1,2,3\})$.

2. yes. For example $f \in (P(\mathbb{N}) \rightsquigarrow \mathbb{N})$ defined $f(C) = x$ if $C = \{x\}$ and undefined otherwise.

3. no. Suppose by contradiction that such f exists, then we can state that also exists $g \in (f(\mathbb{N}) \leftrightarrow P(\mathbb{N})$. The Cantor's theorem states that there is no bijection between a set A and its part $P(A)$, therefore, in particular, there is no bijection between $\mathbb{N}$ and $P(\mathbb{N})$. Since $f(\mathbb{N}) \subseteq \mathbb{N}$, such g cannot actually exists. We reached a contradiction, so f cannot exists.

Question 3. Let A, B be nonempty sets and let $f \in (A \to B)$. Define a function $g \in (B \rightsquigarrow A)$ such that $dom(g) \neq \emptyset$ and for all $b \in dom(g)$, $(f \circ g)(b) = b$.

Answer 3.1. We define $g \in (f(A) \to A)$ for all $b \in f(A)$ as $g(b) = a$ where $f(a) = b$. Since f is a total function, $f(a)$ is always defined. Since $f(a) \in f(A)$, $g(b)$ makes sense. Observe that $dom(g) \neq \emptyset$, because f is a total function and $A \neq \emptyset$, therefore $f(A) \neq \emptyset$ and every element of $f(A)$ has a preimage. For all $b \in dom(g)$, f(g(b))= f(a)=b (see above).