Computability Assignment
Year 2012/13 - Number 4

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1 Question

Let A, B be sets and suppose that $A \leftrightarrow B$ (i.e. there exists a bijection $f \in (A \to B)$). Show that for all sets C, $(C \to (A \times A)) \leftrightarrow (C \to (A \times B))$.

1.1 Answer

In order to have a bijection between $C \to (A \times A)$ and $C \to (A \times B)$ we have to find a bijective function $g \in (A \to B)$ needed to transform the second value a_2 of the pair $(a_1, a_2) \in (A \times A)$ into a $b \in B$ — having so a pair $(a_3, b) \in (A \times B)$ — and vice versa — i.e. $g^{-1}(b) = a_2$ —. This function g can just be f because it's a bijection between A and B. Having a bijection between A and A itself it's trivial, and for C there are no problems, so at this point we can surely affirm that $\forall C.(C \to (A \times A)) \leftrightarrow (C \to (A \times B))$.

2 Question

1. Does a surjective function $f \in (\mathbb{N} \to (\mathbb{N} \to \{0, 1, 2, 3\}))$ exist?
2. Does an injective function $f \in (\mathcal{P}(\mathbb{N}) \rightsquigarrow \mathbb{N})$ exist?
3. Does an injective function $f \in (\mathcal{P}(\mathbb{N}) \to \mathbb{N})$ exist?

Justify your answers.

2.1 Answer

1. It cannot exist a such function because the set of all the functions $\mathbb{N} \to \{0, 1, 2, 3\}$ has a cardinality greater than the set of naturals $\mathbb{N}$, so it is not possible to cover all the (infinite) functions with the (infinite) natural numbers since the it's an infinite of a greater order than $\mathbb{N}$. In order to do a such thing it should be possible to have $f(x)$ that returns more than one result.

2. It exists; for instance we can take $f(\{x\}) = x$ which take only all the partitions containing only one element, mapping that set $\{x\}$ into x. We can do this work because f is partial.

3. It cannot exist because, first of all, having f total we have to map every input — i.e. every $x \in \mathcal{P}(\mathbb{N})$ — , and moreover the cardinality of $\mathcal{P}(\mathbb{N})$ is greater than the$\mathbb{N}$'s one, and finally the injectivity obliges us to map for each different x a different $f(x)$. This means that we haven't got sufficient (infinite) values in $\mathbb{N}$ to map all the ("more" infinite) partitions in $\mathcal{P}(\mathbb{N})$.

3 Question

Let A, B be nonempty sets and let $f \in (A \to B)$. Define a function $g \in (B \rightsquigarrow A)$ such that $\text{dom}(g) \neq \emptyset$ and for all $b \in \text{dom(g)}$, $(f \circ g)(b) = b$.

3.1 Answer

Since g is partial, it is not defined for all the values of B, so $\text{dom}(g) \subset B$. At this point we can take $g = f^{-1}$ to easily reach the goal. Note that $g = f^{-1} \implies \text{ran}(f) = \text{dom}(g) \subset B$. So $\forall b \in \text{dom}(g).\ f(g(b)) = f(f^{-1}(b)) = b$, while $\forall b' \notin \text{dom}(g).\ f(f^{-1}(b'))$ is not defined.