Computability Assignment
Year 2012/13 - Number 4

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1 Question

Let A, B be sets and suppose that $A \leftrightarrow B$ (i.e. there exists a bijection $f \in (A \to B)$). Show that for all sets C, $(C \to (A \times A)) \leftrightarrow (C \to (A \times B))$.

1.1 Answer

let $g \in ((C \to (A \times A)) \to (C \to (A \times B)))$, $g(\langle c, \langle a, a' \rangle \rangle) = \langle c, \langle a, f(a') \rangle \rangle$and
let $h \in ((C \to (A \times B)) \to (C \to (A \times A)))$, $h(\langle c, \langle a, b \rangle \rangle) = \langle c, \langle a, f^{-1}(b) \rangle \rangle$.

Note that $h \circ g(\langle c, \langle a, a' \rangle \rangle) = h(\langle c, \langle a, f(a') \rangle \rangle) = \langle c, \langle a, f^{-1}(f(a')) \rangle \rangle = \langle c, \langle a, a' \rangle \rangle$, thus $h \circ g = id$;

dually, $g \circ h(\langle c, \langle a, b \rangle \rangle) = g(\langle c, \langle a, f^{-1}(b) \rangle \rangle) = \langle c, \langle a, f(f^{-1}(b)) \rangle \rangle = \langle c, \langle a, b \rangle \rangle$, thus $g \circ h = id$.

Thus g is invertible (and its inverse is h), thus g is a bijection.

2 Question

1. Doeas a surjective function $f \in (\mathbb{N} \to (\mathbb{N} \to \{0, 1, 2, 3\}))$ exist?
2. Does an injective function $f \in (\mathcal{P}(\mathbb{N}) \rightsquigarrow \mathbb{N})$ exist?
3. Does an injective function $f \in (\mathcal{P}(\mathbb{N}) \to \mathbb{N})$ exist?

Justify your answers.

2.1 Answer

1. No. Suppose such f exists and let $g(n) = (f(n)(n)+1)\ mod\ 4$ (where mod is the reminder of the integer division). Clearly, $g \in (\mathbb{N} \to \{0, 1, 2, 3\})$; because f is surjective, $\exists m \in \mathbb{N}.\ g = f(m)$; this means that, for some m, $\forall x \in \mathbb{N}.\ f(m)(x) = g(x)$, thus $f(m)(m) = g(m) = (f(m)(m)+1)\ mod\ 4 \neq f(m)(m)$, which is a contraddiction.

2. Yes: consider $f(\{x\}) = x$ with $x \in \mathbb{N}$; $f(\emptyset)$undefined; $f(A)$ undefined if$|A| > 1$. This is clearly a partial function from $\mathcal{P}(\mathbb{N})$ to $\mathbb{N}$ and it is injective (no two distinct elements of the domain are mapped to the same element).

3. No. Suppose such f exists and let $A = ran(f)$; then $A \subseteq \mathbb{N}$ and note $f' \in (\mathcal{P}(\mathbb{N}) \to A)$is bijective; then consider its inverse f'^{-1}(which does exist) and note $ran(f'^{-1}) = \mathcal{P}(\mathbb{N})$. Note that A can not be finite, because otherways $ran(f'^{-1}) = f'^{-1}(A)$ would be finite as well; however, because $A \subseteq \mathbb{N}$, and A not finite, A is an enumerable set i.e. $A \leftrightarrow \mathbb{N}$. So we found $\mathcal{P}(\mathbb{N}) \leftrightarrow A \leftrightarrow \mathbb{N}$, thus $\mathcal{P}(\mathbb{N}) \leftrightarrow \mathbb{N}$, which contraddicts Cantor's theorem.

3 Question

Let A, B be nonempty sets and let $f \in (A \to B)$. Define a function $g \in (B \rightsquigarrow A)$ such that $\text{dom}(g) \neq \emptyset$ and for all $b \in \text{dom(g)}$, $(f \circ g)(b) = b$.

3.1 Answer

Let $C = ran(f) = \{b \in B | \exists a \in A.\ f(a) = b\}$. Note that $C \neq \emptyset$ because f is defined on every element of the nonempty A (so it is defined on at least one number). Note that, by definition of C, $\forall c \in C.\ \exists a \in A.\ f(a) = c$, thus define $g \in (C \to A)$ as a function that coherently returns one such element a of A that satisfies $f(a) = c$ where c is its input. Thus $g \in (B \rightsquigarrow A)$, $\text{dom}(g) = C \neq \emptyset$, and the last requirement is also satified by construction.