Computability Assignment
Year 2012/13 - Number 3

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1 Question

Recall the notions of image and preimage of a set with respect to a function: formally, if $A \subseteq X$, then $f(A) = \{f(x)|x \in A\} \subseteq Y$ and that, if $B \subseteq Y$, then $f^{-1}(B) = \{x|x \in X \wedge f(x) \in B\} \subseteq X$. (Note that here A and B are not points in the domains of f, f^{-1}, but rather sets of such points)

1. For $A \subseteq X$, determine the relation $(\subseteq, =, \supseteq)$ between A and $f^{-1}(f(A))$.
2. For $B \subseteq Y$, determine the relation $(\subseteq, =, \supseteq)$ between B and $f(f^{-1}(B))$.
3. If $C \subset A \subseteq X$, is it always true that $f(C) \subset f(A)$?
4. If $C \subset B \subseteq Y$ and $f^{-1}(B) \neq \emptyset$, is it always true that $f^{-1}(C) \subset f^{-1}(B)$?

1.1 Answer

1. $f(A)$ is the set $f(A) = \{f(x)|x \in A\}$. Applying f^{-1} we get $f^{-1}(\{f(x)|x \in A\}) = \{x|x \in X \wedge f(x) \in \{f(x)|x \in A\}\}$. It is the set of elements starting from X that goes only into images of element of A. Having no information about f, e.g its injectivity, we can only conclude that $A \subseteq f^{-1}(f(A))$.
2. $f^{-1}(B) = \{x|x \in X \wedge f(x) \in B\}$ applying $f()$ we get $f(f^{-1}(B)) = \{f(x)|x \in \{z|z \in X \wedge f(z) \in B\}\} = \{f(x)|x \in X \wedge f(x) \in B\}\}$. Taking X as the domain of f, there could be elements inside B that are not image of f so we can only conclude that $B \supseteq f(f^{-1}(B))$.
3. $f(C) = \{f(x)|x \in C\}$ and $f(A) = \{f(x)|x \in A\}$.We have no information about the behaviour of f, e.g. its injectivity but, if it is not injective,

there could be the case that $f(C) = f(A)$ so it not always true that $f(C) \subset f(A)$.

4. It is not always true. We take $f^{-1}(C) = \{x | x \in X \wedge f(x) \in C\}$ and $f^{-1}(B) = \{x | x \in X \wedge f(x) \in B\}$. As 3) we have no information about f, e.g. its surjectivity, so there could be the case that all the elements of X applied to f brings to C. This would be the case that all the hypothesis hold and $f^{-1}(C) = f^{-1}(A)$.

2 Question

Let A, B be sets, and let $\mathsf{id}_A, \mathsf{id}_B$ denote the identity functions over A and B respectively. Assume $f \in (A \to B)$ and $g \in (B \to A)$ be functions satisfying $g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$, where as usual $\circ$ denotes function composition. Prove that f is a bijection (i.e., injective and surjective).

2.1 Answer

We know that f and g are functions, so $\forall x.x \in A, \exists f(x) \wedge f(x) \in B$ and $\forall y.y \in B, \exists g(y) \wedge g(y) \in A$.

Knowing that $g \circ f = \mathsf{id}_A$, means that f is injective as it is never the case that it maps multiple element of A into the same element of B, otherwise it would not be possible to construct the function id_A. Knowing also that $f \circ g = \mathsf{id}_B$ means that it is surjective as it has to map and cover all the elements of B, otherwise it would not be possible to construct the function id_B. So we can say that f is a bijection.

3 Question

(This question is more challenging.) Find two functions $f, g \in (\mathbb{N} \to \mathbb{N})$ that satisfy all the following conditions:

1. $\mathsf{ran}(f) \neq \mathbb{N}$ and $\mathsf{ran}(g) \neq \mathbb{N}$;
2. $\mathsf{ran}(f)$ and $\mathsf{ran}(g)$ are infinite sets;
3. $\mathsf{ran}(h) = \mathbb{N}$ where $h(n) = f(n) + g(n)$;
4. $\exists n \in \mathbb{N}.\ \mathsf{ran}(g \circ f) = \{n\}$.

3.1 Answer

$f(x) = \begin{cases} id & if\ x\ even \\ 0 & if\ x\ odd \end{cases}$ and $g(x) = \begin{cases} 0 & if\ x\ even \\ id & if\ x\ odd \end{cases}$

They have both infinite range that not covers all $\mathbb{N}$. Summed with the same input they return id+0 or 0+id, so they behave like the identity and composed they always return a costant equal to zero;