Computability Assignment
Year 2012/13 - Number 3

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1 Question

Recall the notions of image and preimage of a set with respect to a function: formally, if $A \subseteq X$, then $f(A) = \{f(x)|x \in A\} \subseteq Y$ and that, if $B \subseteq Y$, then $f^{-1}(B) = \{x|x \in X \wedge f(x) \in B\} \subseteq X$. (Note that here A and B are not points in the domains of f, f^{-1}, but rather sets of such points)

1. For $A \subseteq X$, determine the relation $(\subseteq, =, \supseteq)$ between A and $f^{-1}(f(A))$.
2. For $B \subseteq Y$, determine the relation $(\subseteq, =, \supseteq)$ between B and $f(f^{-1}(B))$.
3. If $C \subset A \subseteq X$, is it always true that $f(C) \subset f(A)$?
4. If $C \subset B \subseteq Y$ and $f^{-1}(B) \neq \emptyset$, is it always true that $f^{-1}(C) \subset f^{-1}(B)$?

1.1 Answer

1. Since we don't have any assumption of the totality of f, the set generated by $f(A)$ may only contain the less or equal elements than the ones that are in A. Since f^{-1}also may only generate a number of elements less than or equal than the one specified, we can conclude that $f^{-1}(f(A))$ is only a subset of A. We cannot say that $A \subseteq f^{-1}(f(A))$ because it would imply that f or f^{-1}generated more elements than their input (which is false for a proper function).
2. This is the symmetric case with respect to point 1. $B \supseteq f(f^{-1}(B))$.
3. This is false, A counterexample is this: suppose f is defined for all points in C, and undefined otherwise. Now $C \subset A$, so A will also contain all

points of C, and $f(C) \subseteq f(A)$. Now, since f is defined only for points in C, $f(A)$ will only contain the results of the points in C, so $f(A) = f(C)$, In this case the proposition $f(C) \subset f(A)$ is not valid, since the two sets are equal.

4. See the demonstration for point 3, substituting f with f^{-1}. Simply take f to be defined for all points in C and undefined otherwise.

2 Question

Let A, B be sets, and let $\mathsf{id}_A, \mathsf{id}_B$ denote the identity functions over A and B respectively. Assume $f \in (A \to B)$ and $g \in (B \to A)$ be functions satisfying $g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$, where as usual $\circ$ denotes function composition. Prove that f is a bijection (i.e., injective and surjective).

2.1 Answer

To be a bijection, a function needs to be injective and surjective. Let's prove that f is a bijection:

- **Injection**: Suppose that f is not injective. This means that that there exist at least a point x that is the result of $f(a)$ and $f(b)$. if we apply g to x, we can only obtain a or b, but not both. This means that $g \circ f \neq \mathsf{id}_A$, which is a contradiction. f is injective.
- **Surjection**: Suppose that f is not surjective, so that $\exists x \in B$ such that $\nexists y \,.\, f(y) = x$. Now, by assumption $f \circ g = \mathsf{id}_B$, but this is impossible, because there would exist an element in Bthat cannot be an output of f. f is surjective.

Since f is surjective and injective, it is also bijective. Since the definitions are symmetric, we can prove using the same procedure that g is also bijective.

3 Question

(This question is more challenging.) Find two functions $f, g \in (\mathbb{N} \to \mathbb{N})$ that satisfy all the following conditions:

1. $\mathsf{ran}(f) \neq \mathbb{N}$ and $\mathsf{ran}(g) \neq \mathbb{N}$;
2. $\mathsf{ran}(f)$ and $\mathsf{ran}(g)$ are infinite sets;
3. $\mathsf{ran}(h) = \mathbb{N}$ where $h(n) = f(n) + g(n)$;
4. $\exists n \in \mathbb{N}.\ \mathsf{ran}(g \circ f) = \{n\}$.

3.1 Answer