Computability Assignment
Year 2012/13 - Number 3

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1 Question

Recall the notions of image and preimage of a set with respect to a function: formally, if $A \subseteq X$, then $f(A) = \{f(x)|x \in A\} \subseteq Y$ and that, if $B \subseteq Y$, then $f^{-1}(B) = \{x|x \in X \wedge f(x) \in B\} \subseteq X$. (Note that here A and B are not points in the domains of f, f^{-1}, but rather sets of such points)

1. For $A \subseteq X$, determine the relation $(\subseteq, =, \supseteq)$ between A and $f^{-1}(f(A))$.
2. For $B \subseteq Y$, determine the relation $(\subseteq, =, \supseteq)$ between B and $f(f^{-1}(B))$.
3. If $C \subset A \subseteq X$, is it always true that $f(C) \subset f(A)$?
4. If $C \subset B \subseteq Y$ and $f^{-1}(B) \neq \emptyset$, is it always true that $f^{-1}(C) \subset f^{-1}(B)$?

1.1 Answer

1. $=$
2. $=$
3. No (since f might not be injective, thus the elements belonging only to A could have the same images of the elements of C, proper subset of A, for example it could send all inputs in 1).
4. No (since f might not be surjective, and while the pre-image of B contains all the elements in the pre-image of C, because their images, the elements of C, are a proper subset of B, it might be that the elements of B that are not in C are not actually the image of any element of X, therefore the pre-image of B and C might be identical).

2 Question

Let A, B be sets, and let $\mathsf{id}_A, \mathsf{id}_B$ denote the identity functions over A and B respectively. Assume $f \in (A \to B)$ and $g \in (B \to A)$ be functions satisfying $g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$, where as usual $\circ$ denotes function composition. Prove that f is a bijection (i.e., injective and surjective).

2.1 Answer

Suppose that f is not injective. Then $\exists a_1, a_2 \in A. f(a_1) = f(a_2) = b, b \in B$. But, since$g \circ f = \mathsf{id}_A$, $a_1 = g(f(a_1)) = g(b) = g(f(a_2))$, contradiction. Thus, f is injective. Then, suppose that f is not surjective. Then $\exists b \in B. \forall a \in A. f(a) \neq b$. But, since $f \circ g = \mathsf{id}_B$, $\forall b \in B. b = f(g(b)) = f(a)$ with$a \in A$, contradiction. Therefore f is surjective, and, being also injective, a bijection.□

3 Question

(This question is more challenging.) Find two functions $f, g \in (\mathbb{N} \to \mathbb{N})$ that satisfy all the following conditions:

1. $\mathsf{ran}(f) \neq \mathbb{N}$ and $\mathsf{ran}(g) \neq \mathbb{N}$;
2. $\mathsf{ran}(f)$ and $\mathsf{ran}(g)$ are infinite sets;
3. $\mathsf{ran}(h) = \mathbb{N}$ where $h(n) = f(n) + g(n)$;
4. $\exists n \in \mathbb{N}.\ \mathsf{ran}(g \circ f) = \{n\}$.

3.1 Answer

$$g(x) = \begin{cases} 0 & x = 2n, n \in \mathbb{N} \\ x & o.w. \end{cases}$$

$$f(x) = \begin{cases} 0 & x = 2n+1, n \in \mathbb{N} \\ x & o.w. \end{cases}$$