Computability Assignment
Year 2012/13 - Number 3

Please keep this file anonymous: do not write your name inside this file.

More information about assignments at http://disi.unitn.it/∼zunino/teaching/computability/assignments

Please do not submit a file containing only the answers; edit this file, instead, filling the answer sections.

1 Question

Recall the notions of image and preimage of a set with respect to a function: formally, if $A \subseteq X$, then $f(A) = \{f(x)|x \in A\} \subseteq Y$ and that, if $B \subseteq Y$, then $f^{-1}(B) = \{x|x \in X \wedge f(x) \in B\} \subseteq X$. (Note that here A and B are not points in the domains of f, f^{-1}, but rather sets of such points)

1. For $A \subseteq X$, determine the relation $(\subseteq, =, \supseteq)$ between A and $f^{-1}(f(A))$.
2. For $B \subseteq Y$, determine the relation $(\subseteq, =, \supseteq)$ between B and $f(f^{-1}(B))$.
3. If $C \subset A \subseteq X$, is it always true that $f(C) \subset f(A)$?
4. If $C \subset B \subseteq Y$ and $f^{-1}(B) \neq \emptyset$, is it always true that $f^{-1}(C) \subset f^{-1}(B)$?

1.1 Answer

1. $f^{-1}(f(A)) = f^{-1}(\{f(x)|x \in A\}) = \{x|x \in X \wedge f(x) \in \{f(x)|x \in A\}\} = \{x|x \in X \wedge x \in A\} = A$;
2. $f(f^{-1}(B)) = f(\{x|x \in X \wedge f(x) \in B\}) = f(\{x|x \in X \wedge f(x) \in \{f(x)|x \in A\}\}) = f(\{x|x \in X \wedge f(x) \in \{f(x)|x \in \{x|x \in X \wedge f(x) \in B\}\}\}) = \{f(x)|x \in X \wedge \{f(x)|x \in \{x|x \in X \wedge f(x) \in B\}\}\} = \{f(x)|x \in X \wedge \{f(x)|x \in X \wedge f(x) \in B\} = \{f(x)x \in X \wedge \{f(x)|x \in X \wedge x \in B\}\} = \{f(x)|x \in X \wedge x \in B\} = \{x|x \in X \wedge x \in B\} = \{x|x \in B\} = B$;
3. Yes, because $f(C) = \{f(x)|x \in C\}$, $f(A) = \{f(x)|x \in A\}$ and since$C \subset A$ then $f(C) \subset f(A)$;
4. Yes, because $f^{-1}(C) = \{x|x \in X \wedge f(x) \in C\}$, $f^{-1}(B) = \{x|x \in X \wedge f(x) \in B\}$ and since $C \subset B$ then $f^{-1}(C) \subset f^{-1}(B)$.

2 Question

Let A, B be sets, and let $\mathsf{id}_A, \mathsf{id}_B$ denote the identity functions over A and B respectively. Assume $f \in (A \to B)$ and $g \in (B \to A)$ be functions satisfying $g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$, where as usual $\circ$ denotes function composition. Prove that f is a bijection (i.e., injective and surjective).

2.1 Answer

- injective: take $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$ and we want to show that $a_1 = a_2$. By definition we have that $a_1 = id_A(a_1) = g \circ f(a_1) = g(f(a_1)) = g(f(a_2)) = g \circ f(a_2) = id_A(a_2) = a_2$. Then we can say that f is injective;
- surjective: take $b_1 \in B$ and we want to show that $\exists a_1 \in A.f(a_1) = b_1$. By definition $b_1 = id_B(b_1) = f \circ g(b_1) = f(g(b_1))$. But $g \in (B \longrightarrow A)$ and this means that $\forall b \in B.\exists! a \in A.g(b) = a$. We call $a_1 = g(b_1)$ and than we can say that f is surjective;
- Since f is injective and surjective by definition we can say f is bijective.

3 Question

(This question is more challenging.) Find two functions $f, g \in (\mathbb{N} \to \mathbb{N})$ that satisfy all the following conditions:

1. $\mathsf{ran}(f) \neq \mathbb{N}$ and $\mathsf{ran}(g) \neq \mathbb{N}$;
2. $\mathsf{ran}(f)$ and $\mathsf{ran}(g)$ are infinite sets;
3. $\mathsf{ran}(h) = \mathbb{N}$ where $h(n) = f(n) + g(n)$;
4. $\exists n \in \mathbb{N}.\ \mathsf{ran}(g \circ f) = \{n\}$.

3.1 Answer

We define:

$$f(n) = \begin{cases} n\ if\ n\ is\ even \\ 0\ if\ n\ is\ odd \end{cases} ;\ g(n) = \begin{cases} n\ if\ n\ is\ odd \\ 0\ if\ n\ is\ even \end{cases}$$

Now we check whether the two functions satisfy the above conditions:

1. By definition $ran(f) = \{n \in \mathbb{N} | n\ is\ even\} \neq \mathbb{N}$ e $ran(g) = \{0\} \cup \{n \in \mathbb{N} | n\ is\ odd\} \neq \mathbb{N}$.
2. It is a consequence of point 1.
3. If n is even allora $h(n) = n + 0 = n$ and if n is odd $h(n) = 0 + n = n$. This means that $h = id_{\mathbb{N}}$ which implies that $ran(h) = \mathbb{N}$.

4. $g \circ f(n) = g(f(n))$. If we find that $\forall n \in \mathbb{N}. g \circ f(n) = 0$ than $ran(g \circ f) = \{0\}$. We prove:

- if n is even $g(f(n)) = g(n) = 0$;
- if n is odd $g(f(n)) = g(0) = 0$.