Computability Assignment
Year 2012/13 - Number 3

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1 Question

Recall the notions of image and preimage of a set with respect to a function: formally, if $A \subseteq X$, then $f(A) = \{f(x)|x \in A\} \subseteq Y$ and that, if $B \subseteq Y$, then $f^{-1}(B) = \{x|x \in X \wedge f(x) \in B\} \subseteq X$. (Note that here A and B are not points in the domains of f, f^{-1}, but rather sets of such points)

1. For $A \subseteq X$, determine the relation $(\subseteq, =, \supseteq)$ between A and $f^{-1}(f(A))$.
2. For $B \subseteq Y$, determine the relation $(\subseteq, =, \supseteq)$ between B and $f(f^{-1}(B))$.
3. If $C \subset A \subseteq X$, is it always true that $f(C) \subset f(A)$?
4. If $C \subset B \subseteq Y$ and $f^{-1}(B) \neq \emptyset$, is it always true that $f^{-1}(C) \subset f^{-1}(B)$?

1.1 Answer

1. It depends on f:
   (a) if f is an injective function A $= f^{-1}(f(A))$
   (b) otherwise A $\supseteq f^{-1}(f(A))$
2. It depends on f:
   (a) if f is a surjective function B $= f^{-1}(f(B))$
   (b) otherwise B $\supseteq f^{-1}(f(B))$
3. No, because if $f(A \setminus C) = \emptyset$ then $f(A) = f(C)$
4. No, because if $f^{-1}(B \setminus C) = \emptyset$ then $f^{-1}(B) = f^{-1}(C)$

2 Question

Let A, B be sets, and let $\mathsf{id}_A, \mathsf{id}_B$ denote the identity functions over A and B respectively. Assume $f \in (A \to B)$ and $g \in (B \to A)$ be functions satisfying $g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$, where as usual $\circ$ denotes function composition. Prove that f is a bijection (i.e., injective and surjective).

2.1 Answer

1. Proof of injectivity of f.
   We assume that: $\forall x \in A.x = g(f(x))$
   Suppose that $\exists xy.f(x) = f(y) \wedge x \neq y$
   then $x = g(f(x)) = g(f(y)) = y$ because $f(x) = f(y)$ and the function g with the same argument returns the same value: $x = y$.
   But we suppose that $x \neq y$. Contradiction.
   Therefore $\neg\exists xy.f(x) = f(y) \wedge x \neq y$
   $\forall xy.\neg((f(x) = f(y)) \wedge \neg(x = y))$
   $\forall xy.(\neg(f(x) = f(y)) \vee (x = y))$
   $\forall xy.(f(x) = f(y)) \Rightarrow (x = y)$
   So, to compose the functions g and f in order to have I_A it is necessary the injectivity of f.

2. Proof of surjectivity of f.
   $f \circ g = I_B \Rightarrow \forall y \in B.\exists x \in A.(x = g(y) \wedge f(x) = y)$
   A function f is surjective if $\forall y \in B.\exists x \in A.f(x) = y$
   We prove by contradiction that if f in not surjective
   $\neg\forall y \in B.\exists x \in A.f(x) = y$
   $\exists y \in B.\forall x \in A.\neg(f(x) = y)$
   $\exists y \in B.\forall x \in A.\neg f(g(y)) = y$
   But $f \circ g = I_B \Leftrightarrow \forall y \in B.\exists x \in A.(x = g(y) \wedge f(x) = y)$. Contradiction.

3. If f is injective and surjective, as proved, f is bijective.

3 Question

(This question is more challenging.) Find two functions $f, g \in (\mathbb{N} \to \mathbb{N})$ that satisfy all the following conditions:

1. $\mathsf{ran}(f) \neq \mathbb{N}$ and $\mathsf{ran}(g) \neq \mathbb{N}$;
2. $\mathsf{ran}(f)$ and $\mathsf{ran}(g)$ are infinite sets;
3. $\mathsf{ran}(h) = \mathbb{N}$ where $h(n) = f(n) + g(n)$;
4. $\exists n \in \mathbb{N}.\ \mathsf{ran}(g \circ f) = \{n\}$.

3.1 Answer

$f(x) = \begin{cases} x & \exists n \in \mathbb{N}. 2n = x \\ 0 & o.w. \end{cases}$ $g(x) = \begin{cases} x & \exists n \in \mathbb{N}. 2n + 1 = x \\ 0 & o.w. \end{cases}$

1. ran(f) is all the even natural numbers and ran(g) is all the odd natural numbers, but their ran is different from $\mathbb{N}$
2. ran(f) and ran(g) are both infinite
3. $\mathsf{ran}(h) = \mathbb{N}$ beacause is the union of all even and odd natural numbers
4. The number n is 0