Computability Assignment
Year 2012/13 - Number 3

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1 Question

Recall the notions of image and preimage of a set with respect to a function: formally, if $A \subseteq X$, then $f(A) = \{f(x)|x \in A\} \subseteq Y$ and that, if $B \subseteq Y$, then $f^{-1}(B) = \{x|x \in X \wedge f(x) \in B\} \subseteq X$. (Note that here A and B are not points in the domains of f, f^{-1}, but rather sets of such points)

1. For $A \subseteq X$, determine the relation $(\subseteq, =, \supseteq)$ between A and $f^{-1}(f(A))$.
2. For $B \subseteq Y$, determine the relation $(\subseteq, =, \supseteq)$ between B and $f(f^{-1}(B))$.
3. If $C \subset A \subseteq X$, is it always true that $f(C) \subset f(A)$?
4. If $C \subset B \subseteq Y$ and $f^{-1}(B) \neq \emptyset$, is it always true that $f^{-1}(C) \subset f^{-1}(B)$?

1.1 Answer

1. We know that $\forall x \in A.\ f(x) \in Y$. Applying to this $y = f(x)$ the function f^{-1}, we can have two different cases:
   (a) $y \in Y \setminus B \implies f^{-1}(f(x))$ is not defined because $f(x) \notin B$;
   (b) $y \in B \implies f^{-1}(f(x)) \in X$, which means that $z = f^{-1}f(x))$ can belongs either to $X \setminus A$ or A. So we can say that $A \subseteq f^{-1}(f(A))$ because it's possible that $\exists x \in A.\ f^{-1}(f(x)) \in X \setminus A$ (or, in other words, it's not sure that $\forall x \in A.\ f^{-1}(f(x)) \in A$).
2. Following the same reasoning in (1.b), $B \subseteq f(f^{-1}(B))$ because there is the possibility that $\exists y \in B.\ f(f^{-1}(x)) \in Y \setminus B$.

3. No because we haven't any information about the injectivity or surjectivity of f. So we have the following cases:

   (a) $f(C) \subseteq f(A) \subseteq Y \setminus B \subset Y \implies f(C) \not\subset f(A)$

   (b) $f(C) \subseteq f(A) \subseteq B \subset Y \implies f(C) \not\subset f(A)$

   (c) $f(C) \subset f(A) \subset B$

4. Again, not necessarily because we have no information about the injectivity of f^{-1}, so we could have the case where $f^{-1}(C) = f^{-1}(B)$.

2 Question

Let A, B be sets, and let $\mathsf{id}_A, \mathsf{id}_B$ denote the identity functions over A and B respectively. Assume $f \in (A \to B)$ and $g \in (B \to A)$ be functions satisfying $g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$, where as usual $\circ$ denotes function composition. Prove that f is a bijection (i.e., injective and surjective).

2.1 Answer

Let's prove the injectivity (1.) and surjectivity (2.) of f.

1. The function f is injective because of this reasoning. $\forall x, y \in A.\ f(x) \neq f(y) \implies x \neq y$. Being $\mathsf{id}_A = g(f(x))$, $\mathsf{id_B} = g(f(y))$ and $f(x) \neq f(y)$, then $g(f(x)) \neq g(f(y)) \implies \mathsf{id}_A(x) \neq \mathsf{id}_B(y) \implies x \neq y$.

2. The function f is also surjective because it's just necessary to apply g after f to go back the same domain's value, since $g \circ f = \mathsf{id}_A$. So $\forall y \in B.\ \exists x \in A \mid g(y) = f^{-1}(y) = x$.

3 Question

(This question is more challenging.) Find two functions $f, g \in (\mathbb{N} \to \mathbb{N})$ that satisfy all the following conditions:

1. $\mathsf{ran}(f) \neq \mathbb{N}$ and $\mathsf{ran}(g) \neq \mathbb{N}$;

2. $\mathsf{ran}(f)$ and $\mathsf{ran}(g)$ are infinite sets;

3. $\mathsf{ran}(h) = \mathbb{N}$ where $h(n) = f(n) + g(n)$;

4. $\exists n \in \mathbb{N}.\ \mathsf{ran}(g \circ f) = \{n\}$.

3.1 Answer