Year 2013/14 - Number 3

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Question 1. Recall the notions of image and preimage of a set with respect to a function: formally, if $A \subseteq X$, then $f(a) = \{f(x)|x \in A\} \subseteq Y$ and that, if $B \subseteq Y$, then $f^{-1}(B) = \{x|x \in X \land f(x) \in B\} \subseteq X$. (Note that here A and B are not points in the domains of f, f^{-1}, but rather sets of such points)

1. For $A \subseteq X$, determine the relation $(\subseteq, =, \supseteq)$ between A and $f^{-1}(f(A))$.

2. For $B \subseteq Y$, determine the relation $(\subseteq, =, \supseteq)$ between B and $f(f^{-1}(B))$.

3. If $C \subset A \subseteq X$, is it always true that $f(C) \subset f(A)$?

4. If $C \subset B \subseteq Y$ and $f^{-1}(B) \neq \emptyset$, is it always true that $f^{-1}(C) \subset f^{-1}(B)$?

Answer 1.1.

1. By definition of $f^{-1}(B)$, it follows that $f^{-1}(f(A)) = \{x|x \in X \land f(x) \in f(A)\}$. The condition $f(x) \in f(A)$ is true iff $x \in A$ (by definition of $f(A)$). That's mean that $f^{-1}(f(A)) = \{x|x \in X \land x \in A\} = \{x|x \in A\} = A$.

2. By definition of $f(A)$, it follows that $f(f^{-1}(B)) = \{f(x)|x \in f^{-1}(B)\}$. The condition $x \in f^{-1}(B)$ is true iff $x \in X \land f(x) \in B$ (by definition of $f^{-1}(B)$). That's mean that $f(f^{-1}(B)) = \{f(x)|x \in X \land f(x) \in B\} = \{x|f(x) \in B\} = B$.

3. If $C \subset A \subseteq X$ is not always true that $f(C) \subset f(A)$. Suppose that $\forall x \in A\ f(x) = p \in B$, then $f(C) = f(A) = \{p\}$.

4. If $C \subset B \subseteq Y$ and $f^{-1}(B) \neq \emptyset$, then it is always true that $f^{-1}(C) \subset f^{-1}(B)$. We prove this statement by contradiction. $C \subset B \Rightarrow \exists x \in B \setminus C$. Certainly, $f^{-1}(x) \in f^{-1}(B)$ and $f^{-1}(C) \subseteq f^{-1}(B)$ because, by definition of function, $\forall x \in A. \exists! y \in B. f(x) = y$. By contradiction, suppose that $f^{-1}(C) =) f^{-1}(B)$. It follows that $\exists y \in C. f^{-1}(y) = f^{-1}(x) \Rightarrow \exists a = f^{-1}(y) \in A. f(a) = x = y \wedge x \neq y$. We reach an absurd situation.

Question 2. Let A, B be sets, and let id_A, id_B denote the identity functions over A and B respectively. Assume $f \in (A \to B)$ and $g \in (B \to A)$ be functions satisfying $g \circ f = id_A$ and $f \circ g = id_B$, where as usual $\circ$ denotes function composition. Prove that f is a bijection (i.e., injective and surjective).

Answer 2.1. We prove that
f is surjective: f is surjective $\Leftrightarrow \forall y \in B. \exists x \in A$ such that $f(x) = y$. Let $y \in B$. g is a total function, so $g(y)$ is defined. In particular $g(y) \in A$. f is a total function, so $f(g(y))$ is defined. By hypothesis, $f \circ g = id_B \Rightarrow f(g(y)) = id_B(y) = y$. So $\forall y \in B. \exists x = g(y) \in A$ such that $f(x) = y$.

f is injective: f is injective $\Leftrightarrow f(x) = f(x') \to x = x'$. Let $x \in A$. f is a total function so $f(x)$ is defined. In particular $f(x) \in B$. g is a total function, so $g(f(x))$ is defined. By hypothesis, $g \circ f = id_A \Rightarrow g(f(x)) = id_A(x) = x$. Similarly, $g(f(x')) = x'$. Suppose that $f(x) = f(x')$, then $x = g(f(x)) = g(f(x')) = x' \Rightarrow x = x'$.

Question 3. (This question is more challenging.) Find two functions $f, g \in (\mathbb{N} \to \mathbb{N})$ that satisfy all the following conditions:

1. $ran(f) \neq \mathbb{N}$ and $ran(g) \neq \mathbb{N}$;
2. $ran(f)$ and $ran(g)$ are infinite sets;
3. $ran(h) = \mathbb{N}$ where $h(n) = f(n) + g(n)$;
4. $\exists n \in \mathbb{N}. ran(g \circ f) = \{n\}$.

Answer 3.1.

1. Let $g \in (\mathbb{N} \to \mathbb{N})$ be $g(x) = 0$ if $x = 2n + 1$ for some $n \in \mathbb{N}$, $x + 1$ otherwise. It follows that $ran(g) = \{x \in \mathbb{N} | x = 0 \vee x$ is an odd number$\} \neq \mathbb{N}$. Let $f \in (\mathbb{N} \to \mathbb{N})$ be $f(x) = x$ if $x = 2n + 1$ for some $n \in \mathbb{N}$, $x + 1$ otherwise. It follows that $ran(f) = \{x \in \mathbb{N} | x$ is an odd number $\} \neq \mathbb{N}$.
2. $ran(f)$ is an infinite set because it contains all the natural odd numbers, while $ran(g)$ is an infinite set because it's actually the set of all the natural numbers.

3. We prove that the function $h \in (\mathbb{N} \to \mathbb{N})$ defined as $h(n) = f(n)+g(n)$ satisfies the condition $ran(h) = \mathbb{N}$. We can state that $ran(h) \subseteq \mathbb{N}$, so we have to prove that $\mathbb{N} \subseteq ran(h)$. Let $x \in \mathbb{N}$. If x is an odd number, then $g(x) = 0$ and $f(x) = x$, so $h(x) = f(x)+g(x) = x \Rightarrow x \in ran(h)$. If x is an even number, then $x = 2n$ for some $n \in \mathbb{N}$. Let $m \in \mathbb{N}$ be an even number, then $g(m) = m + 1$ and $f(m) = m + 1$, so $h(m) = 2(m+1)$. I can choose $m+1 = n \Rightarrow h(m) = x\ Rightarrowx \in ran(h)$.

4. $\forall x \in \mathbb{N}\ f(n)$ is an odd number. By definition of g, it follows that $g(f(n)) = 0$. We can state that $ran(g \circ f) = 0$.