Computability Assignment
Year 2012/13 - Number 3

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1 Question

Recall the notions of image and preimage of a set with respect to a function: formally, if $A \subseteq X$, then $f(A) = \{f(x)|x \in A\} \subseteq Y$ and that, if $B \subseteq Y$, then $f^{-1}(B) = \{x|x \in X \wedge f(x) \in B\} \subseteq X$. (Note that here A and B are not points in the domains of f, f^{-1}, but rather sets of such points)

1. For $A \subseteq X$, determine the relation $(\subseteq, =, \supseteq)$ between A and $f^{-1}(f(A))$.
2. For $B \subseteq Y$, determine the relation $(\subseteq, =, \supseteq)$ between B and $f(f^{-1}(B))$.
3. If $C \subset A \subseteq X$, is it always true that $f(C) \subset f(A)$?
4. If $C \subset B \subseteq Y$ and $f^{-1}(B) \neq \emptyset$, is it always true that $f^{-1}(C) \subset f^{-1}(B)$?

1.1 Answer

1. $f^{-1}(f(A)) = f^{-1}(\{f(x)|x \in A\}) = \{x|x \in X \wedge f(x) \in \{f(x)|x \in A\}\} = \{x|x \in X \wedge x \in A\} = A$.
2. $f(f^{-1}(B)) = f(\{x|x \in X \wedge f(x) \in B\}) = \{f(x)|x \in \{x|x \in X \wedge f(x) \in B\}\} = \{f(x)|x \in X \wedge f(x) \in B\} = \{y \in B|\exists x \in X.\ f(x) = y\} \subseteq B$.
3. No. For example, let $X = \{0,1\},\ A = X,\ C = \{1\}$and let$f(0) = f(1) = 42$; then $C \subset A \subseteq X$ but still $f(C) = f(A) = \{42\}$.
4. No. For example, let $X = \{0\},\ Y = \{8,9\},\ B = Y,\ C = \{8\},\ f(0) = 8$; then $C \subset B \subseteq Y$ and $f^{-1}(B) = \{0\} \neq \emptyset$, yet $f^{-1}(C) = \{0\} = f^{-1}(B)$

2 Question

Let A, B be sets, and let $\mathsf{id}_A, \mathsf{id}_B$ denote the identity functions over A and B respectively. Assume $f \in (A \to B)$ and $g \in (B \to A)$ be functions satisfying $g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$, where as usual $\circ$ denotes function composition. Prove that f is a bijection (i.e., injective and surjective).

2.1 Answer

- (Injective) By contraddiction, suppose f not injective i.e. $\exists x, z \in A.\ (x \neq y \wedge f(x) = f(z))$; let x, z be such numbers and note that because $f(x) = f(z)$ then$g(f(x)) = g(f(z))$; However, $x \neq z \Rightarrow id_A(x) \neq id_A(z) \Rightarrow g(f(x)) \neq g(f(z))$, which contraddicts the previous statement.

- (Surjective) By contraddiction, suppose f not surjective i.e. $\exists y \in B.\ \nexists x \in A.\ f(x) = y$; let y be such number and let $z = g(y)$ and note $z \in A$. Then by hypothesis $f(z) = f(g(y)) = id_B(y) = y$, but the existance of z contraddicts the initial statement.

3 Question

(This question is more challenging.) Find two functions $f, g \in (\mathbb{N} \to \mathbb{N})$ that satisfy all the following conditions:

1. $\mathsf{ran}(f) \neq \mathbb{N}$ and $\mathsf{ran}(g) \neq \mathbb{N}$;
2. $\mathsf{ran}(f)$ and $\mathsf{ran}(g)$ are infinite sets;
3. $\mathsf{ran}(h) = \mathbb{N}$ where $h(n) = f(n) + g(n)$;
4. $\exists n \in \mathbb{N}.\ \mathsf{ran}(g \circ f) = \{n\}$.

3.1 Answer

Let $f(n) = \begin{cases} n & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$, $g(n) = \begin{cases} 0 & \text{if } n \text{ even} \\ n & \text{if } n \text{ odd} \end{cases}$. Conditions 1 and 2 are obviously satisfied. Note $h = id_{\mathbb{N}}$, thus 3 is also satisfied. Finally, consder $g \circ f$: f maps any natural to an even number and g maps any even to 0, thus $\mathsf{ran}(g \circ f) = \{0\}$.