# Computability Assignment
# Year 2012/13 - Number 3

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## 1 Question

Recall the notions of image and preimage of a set with respect to a function: formally, if $A \subseteq X$, then $f(A) = \{f(x) | x \in A\} \subseteq Y$ and that, if $B \subseteq Y$, then $f^{-1}(B) = \{x | x \in X \wedge f(x) \in B\} \subseteq X$. (Note that here $A$ and $B$ are not points in the domains of $f, f^{-1}$, but rather sets of such points)

1. For $A \subseteq X$, determine the relation $(\subseteq, =, \supseteq)$ between $A$ and $f^{-1}(f(A))$.
2. For $B \subseteq Y$, determine the relation $(\subseteq, =, \supseteq)$ between $B$ and $f(f^{-1}(B))$.
3. If $C \subset A \subseteq X$, is it always true that $f(C) \subset f(A)$?
4. If $C \subset B \subseteq Y$ and $f^{-1}(B) \neq \emptyset$, is it always true that $f^{-1}(C) \subset f^{-1}(B)$?

### 1.1 Answer

1. $A \subseteq f^{-1}(f(A))$: Let's take $a \in A \subseteq X$. Then $f(a) \in f(A)$. Now I have that $f^{-1}(f(A)) = \{x | x \in X \wedge f(x) \in f(A)\}$. Then $a \in f^{-1}(f(A))$. On the contrary, let's call $f(A) = C$. I could take $a' \in X \setminus A$ such that it is mapped by $f$ into an element of $C$. In this case $a' \in X$, $f(a') \in C = f(A)$ then $a' \in f^{-1}(f(A))$ but, by constraction, $a' \notin A$. This case is possibile since $f$, in generale, is not injective.
2. $f(f^{-1}(B)) \subseteq B$: Let's take $b \in f(f^{-1}(B))$. This means that $b \in \{f(x) | x \in f^{-1}(B)\}$. Let's say that $b = f(a)$ for some $a \in X$. Then $a \in f^{-1}(B) = \{x | x \in X \wedge f(x) \in B\}$, which means that $b = f(a) \in B$. On the contrary, Let's take $b \in B$ such that $\nexists a \in A. b = f(a)$. This is possibile since $f$, in general, is not surjective. Then $\forall C \subseteq X. b \notin f(C)$, in particular $b \notin f(f^{-1}(B))$.

3. No: If I take $b \in Y$ and I define a function $f \in (X \to Y)$ such that $\forall a \in A.f(a) = b$, then $f(A) = \{b\} = f(C)$.
4. No: I define a function $f \in (X \to Y)$ such that $\forall b \in B \backslash C.\nexists a \in X.b = f(a)$. Then $f^{-1}(B) = \{x | x \in X \land f(x) \in B\} = \{x | x \in X \land f(x) \in C\} = f^{-1}(C)$.

# 2 Question

Let $A, B$ be sets, and let $\mathsf{id}_A, \mathsf{id}_B$ denote the identity functions over $A$ and $B$ respectively. Assume $f \in (A \to B)$ and $g \in (B \to A)$ be functions satisfying $g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$, where as usual $\circ$ denotes function composition. Prove that $f$ is a bijection (i.e., injective and surjective).

## 2.1 Answer

1. $f$ injective: Let's take $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$ and I want to show that $a_1 = a_2$. $a_1 = id_A(a_1) = g \circ f(a_1) = g(f(a_1)) = g(f(a_2)) = g \circ f(a_2) = id_A(a_2) = a_2$. Then $f$ is injective.
2. $f$ surjective: Let's take $b_0 \in B$, I want to show that $\exists a_0 \in A.f(a_0) = b_0$. $b_0 = id_B(b_0) = f \circ g(b_0) = f(g(b_0))$. But $g \in (B \to A)$ and this means that $\forall b \in B.\exists! a \in A.g(b) = a$. I call $a_0 = g(b_0)$. Then $f$ is surjective.

# 3 Question

(This question is more challenging.) Find two functions $f, g \in (\mathbb{N} \to \mathbb{N})$ that satisfy all the following conditions:

1. $\mathsf{ran}(f) \neq \mathbb{N}$ and $\mathsf{ran}(g) \neq \mathbb{N}$;
2. $\mathsf{ran}(f)$ and $\mathsf{ran}(g)$ are infinite sets;
3. $\mathsf{ran}(h) = \mathbb{N}$ where $h(n) = f(n) + g(n)$;
4. $\exists n \in \mathbb{N}.\ \mathsf{ran}(g \circ f) = \{n\}$.

## 3.1 Answer

Let's define:

$$f(n) = \begin{cases} n\ if\ n\ is\ even \\ 0\ if\ n\ is\ odd \end{cases} \quad ; g(n) = \begin{cases} n\ if\ n\ is\ odd \\ 0\ if\ n\ is\ even \end{cases}$$

Now I proove that theese two functions satisfy all the previous conditions:

1. By definition, $ran(f) = \{n \in \mathbb{N} | n\ is\ even\} \neq \mathbb{N}$ and $ran(g) = \{0\} \cup \{n \in \mathbb{N} | n\ is\ odd\} \neq \mathbb{N}$;
2. It is obvious from 1. ;

3. If $n$ is even then $h(n) = n + 0 = n$, otherwise $h(n) = 0 + n = n$. Then $h = id_{\mathbb{N}}$, which implies that $ran(h) = \mathbb{N}$.

4. $g \circ f(n) = g(f(n))$. If I find that $\forall n \in \mathbb{N}. g \circ f(n) = 0$ then $ran(g \circ f) = \{0\}$. Let's try:

   (a) $n\ even \Rightarrow g(f(n)) = g(n) = 0$

   (b) $n\ odd \Rightarrow g(f(n)) = g(0) = 0$