Computability Assignment
Year 2012/13 - Number 2

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1 Question

In this exercise, $p(x)$ and $q(x)$ will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion);
2. $Q \subset P$ (strict inclusion);
3. $P \setminus Q \neq \emptyset$;
4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

1.1 Answer

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap\{A_i \mid i \in \mathbb{N}\} = \{x \mid \forall i \in \mathbb{N}\ x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cap A_1 \cap \cdots \cap A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

3.1 Answer

1 p_1**:** $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$**;** the hypothesis $(*)$ is sufficient to conclude that p_i holds;

Since we have

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

so that $A_0 \cap A_1 = A_1$, $A_0 \cap A_2 \cap A_3 = A_3$, and

$A_0 \cap A_1 \cap ... \cap A_k = A_k = \bigcap_{i=0}^{k} A_i$ So $p_1$holds

2 p_2: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;

the hypothesis $(*)$ is sufficient to conclude that $p_2$holds

Choose abitrary $j = 0$. Let $A_j = A_0 = \{0,1,2,3\}$, and $A_{j+1} = A_1 = \{0,1,2,3\}$. $A_0$and A_1both satisfies the hypothesis $(*)$, and $A_0 = A_1$. So the $p_2$holds.

3 p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;

the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_3

Looking back into problem 2, we know that there exists $i \in \mathbb{N}$ such that $A_i = A_{i+1}$. So the p_3 holds for this case.

However, there are cases that $p_3$doesn't hold. For example, if we choose a sequence of sets like this

$A_0 = \{0,1,2,3\}$, $A_1 = \{1,2,3\}$. Then it's clearly that $A_0 \neq A_1$.

4 p_4: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
the hypothesis $(*)$ is sufficient to conclude that $p_4$does not hold.

From hypothesis $(*)$ and the condition of p_4, we know that$\forall i \in \mathbb{N}.\ A_{i+1} \subset A_i \subset ... \subset A_0 \subset \mathbb{N}$

And $A_i \neq \emptyset$because if $A_i = \emptyset$, then $A_{i+1} = \emptyset$, which does not satisfy the hypothesis $A_i \neq A_{i+1}$

We can always choose an abitrary $x \in A_{i+1}$ since $A_{i+1} \neq \emptyset$. According to the preliminaries, we have $\bigcap_{i=0}^{\infty} A_i \neq \emptyset$ since $\bigcap_{i=0}^{\infty} A_i$ always have at least 1 element x.

There for $p_4$does not hold.

5 p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
the hypothesis $(*)$ is sufficient to conclude that p_5 holds;
According to the preliminaries
$\bigcap_{i=0}^{\infty} A_i = \{x \mid \forall i \in \mathbb{N}\ x \in A_i\}$

Since $A_i$is finite, $\bigcap_{i=0}^{\infty} A_i$ is a set that contains elements that appear in all set $A_i, \forall i \in \mathbb{N}$, so that we can always count the number of elements in set $\bigcap_{i=0}^{\infty} A_i$.

So the $p_5$holds.

6 p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
the hypothesis $(*)$ is not sufficient to conclude that $p_6$does not hold

7 p_7: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.
the hypothesis $(*)$ is sufficient to conclude that $p_7$holds