Computability Assignment Year 2012/13 - Number 2

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1 Question

In this exercise, p(x) and q(x) will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties p(x) and q(x) such that $\forall x \in \mathbb{N}$. $p(x) \Rightarrow q(x)$ and

- 1. $P \subset Q$ (strict inclusion);
- 2. $Q \subset P$ (strict inclusion);
- 3. $P \setminus Q \neq \emptyset$;
- 4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

1.1 Answer

- 1. p(x) = (x is divisible by 4) and q(x) = (x is even). Then, $\forall x \in \mathbb{N}$. $p(x) \Rightarrow q(x)$ and $P \subset Q$ as P is always smaller than Q.
- 2. It is not possible $\forall x \in \mathbb{N}$. $p(x) \Rightarrow q(x)$ implies that for each element in P, it must also be in Q. So, either P is subset of Q or P is equal to Q.
- 3. It is not possible. From the argument as explained in 2, P is subset of Q. Hence, P/Q is always empty.
- 4. The same example as 1. As set Q is always larger than P, $Q \setminus P \neq \emptyset$

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcap_{i=0}^{\infty}A_i=\bigcap\{A_i\mid i\in\mathbb{N}\}=\{x\mid\forall i\in\mathbb{N}\ x\in A_i\}$ and $\bigcap_{i=0}^kA_i=\bigcap\{A_i\mid i\in\mathbb{N}\ \land\ i\leq k\}=A_0\cap A_1\cap\cdots\cap A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis (*) is sufficient to conclude that p_i holds; or
- the hypothesis (*) is sufficient to conclude that p_i does not hold; or
- the hypothesis (*) is not sufficient to conclude anything about the truth of p_i .

Justify your answers (briefly).

- 1. p_1 : $\forall k \in \mathbb{N}$. $A_k = \bigcap_{i=0}^k A_i$;
- 2. p_2 : if $\forall i \in \mathbb{N}$. A_i is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
- 3. p_3 : for all i, if A_i is finite, then $A_i = A_{i+1}$;
- 4. p_4 : if $\forall i \in \mathbb{N}$. $A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
- 5. p_5 : if $\forall i \in \mathbb{N}$. A_i is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
- 6. p_6 : if $\forall i \in \mathbb{N}$. A_i is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
- 7. p_7 : if $\forall i \in \mathbb{N}$. A_i is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

3.1 Answer

- 1. Sufficient to conclude that p_i holds. As A_k is subset of all the sets A_0 to A_{k-1} , all the elements of A_k should belong to all those sets. It can not have any extra element not belonging to all the sets before A_k
- 2. Sufficient to conclude that p_i holds. Since all the sets A_i are finite, the possible unique subsets of the set is also finite. But there are infinite A_i , hence some of the A_i must repeat. Also, from the condition that $A_{k-1} \supseteq A_k$, only the consecutive sets could be equal. Hence this property is true.

- 3. Insufficient. If $A_i = \{2, 3, 4, 5\}$, then from the given condition we can say $A_i \supseteq A_{i+1}$ which means A_{i+1} can be $\{2, 3, 4, 5\}$ in which case $A_i = A_{i+1}$ is satisfied. But, A_{i+1} can also be $\{2, 3, 4\}$ in which case $A_i = A_{i+1}$ is not true.
- 4. Insufficient.
- 5. Sufficient to conclude that p_i holds. The intersection of finite set is always finite.
- 6. Sufficient to conclude that p_i does not hold. As all sets are infinite and each i+1 th set is subset of i th set, the intersection should be infinite.
- 7. Sufficient to conclude that p_i holds