Computability Assignment
Year 2012/13 - Number 2

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1 Question

In this exercise, $p(x)$ and $q(x)$ will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion);
2. $Q \subset P$ (strict inclusion);
3. $P \setminus Q \neq \emptyset$;
4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

1.1 Answer

1. $p(x) = 2^{x+1}$ $P = \{x \in \mathbb{N} :\ 2^{x+1}\}$
   $q(x) = 2x$ $Q = \{x \in \mathbb{N} :\ 2x\}$
2. it is not possible.
   We proceed by contradiction
   Suppose that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x) \wedge Q \subset P$. This means that if we take any $x \in P$, for the definition of P, $p(x)\ holds$and for $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ we have that$q(x)$holds and $x \in Q$.
   Therefore $\forall x \in \mathbb{N}. x \in P \implies x \in Q$. But this contradicts $Q \subset P$

3. it is not possible.
$P \backslash Q \neq \emptyset \implies \exists p \in P: \ p \notin Q \implies \exists x \in \mathbb{N}.p(x) \wedge \neg q(x) \implies \exists x \in \mathbb{N}.\neg\neg(p(x) \wedge \neg q(x)) \implies \exists x \in \mathbb{N}.\neg(\neg p(x) \wedge q(x)) \implies \exists x \in \mathbb{N}.\neg(p(x) \implies q(x)) \implies \neg \forall x \in \mathbb{N}.p(x) \implies q(x)$ which is a contradiction.

4. $p(x) = x > 2 \quad P = \{x \in \mathbb{N}: \ x > 2\}$
$q(x) = x > 1 \quad Q = \{x \in \mathbb{N}: \ x > 1\}$
$Q \backslash P = \{2\}$

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i \in \mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap\{A_i \mid i \in \mathbb{N}\} = \{x \mid \forall i \in \mathbb{N} \ x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap\{A_i \mid i \in \mathbb{N} \ \wedge \ i \leq k\} = A_0 \cap A_1 \cap \cdots \cap A_k$.

3 Question

Assume $(A_i)_{i \in \mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}. \ A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2: if $\forall i \in \mathbb{N}. \ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4: if $\forall i \in \mathbb{N}. \ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5: if $\forall i \in \mathbb{N}. \ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6: if $\forall i \in \mathbb{N}. \ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7: if $\forall i \in \mathbb{N}. \ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

3.1 Answer

1. the hypothesis $(*)$ is sufficient to conclude that p_1 holds
   We proceed by induction.

   - $k = 0$
     $A_0 = \bigcap_{i=0}^{0} A_i = A_0$
   - inductive step
     Suppose $p_1 holds$for $k \in \mathbb{N}$
     $k = k + 1$
     $A_{k+1} = \bigcap_{i=0}^{k+1} A_i = (\bigcap_{i=0}^{k} A_i) \bigcap A_{k+1} = A_k \bigcap A_{k+1}$
     since $A_k \supseteq A_{k+1} \implies A_k \bigcap A_{k+1} = A_{k+1}$

2. the hypothesis $(*)$ is sufficient to conclude that p_2 holds
   Suppose by contradiction $\neg(\exists j \in \mathbb{N} : \ A_j = A_{j+1}) \implies \forall j \in \mathbb{N}.\ A_j \neq A_{j+1}$
   from this and the assumption $(*)$ we have $\mathbb{N} \supset \cdots \supset A_j \supset A_{j+1} \supset \cdots$
   We have an infinite sequence of finite sets with strictly decreasing cardinality. Since the cardinality of $A_0$is finite, at some point we have a set with 0-cardinality.
   Therefore we will have $A_j = \emptyset$and $A_{j+1} = \emptyset$and thus $A_j = A_{j+1}$which is a contradiction.

3. the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_3.

   - TRUE: $A_0 = \emptyset,\ A_1 = \emptyset$
   - FALSE: $A_0 = \{0, 1, 2, 3\},\ A_1 = \{0, 1, 2\}$

4. the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_4.

   - If $A_i$is finite, $A_i \neq A_{i+1}$is false because of what we have prove in p2. Hence for implication p4 is true.
   - If $A_i$is infinite we find a counter-example that makes p4 false:
     $A_0 = Even \cup \{1\},\ A_1 = Even \cup \{1\} \backslash \{0\},\ A_2 = Even \cup \{1\} \backslash \{0, 2\}$,...
     $\bigcap_{i=0}^{\infty} A_i = 1$

5. the hypothesis $(*)$ is sufficient to conclude that p_5 holds
   We have an infinite sequence of finite sets with strictly decreasing cardinality. Since the cardinality of $A_0$is finite, at some point we have a set with 0-cardinality.
   Therefore the resulting intersection will be an empty set which is finite.

6. the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_6

- FALSE: if we take $\mathbb{N} \supseteq A_0 = A_1 = A_2 = ... \implies \bigcap_{i=0}^{\infty} A_i = A_0 = A_1 = ...$ is infinite
- TRUE: if if we take $\mathbb{N} \supset A_0 \supset A_1 \supset A_2 \supset ... \implies \bigcap_{i=0}^{\infty} A_i$ is finite

7. the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_7

   - TRUE: if we take $\mathbb{N} \supseteq A_0 = A_1 = A_2 = ... \implies \bigcap_{i=0}^{\infty} A_i = A_0 = A_1 = ...$ is infinite
   - FALSE: if if we take $\mathbb{N} \supset A_0 \supset A_1 \supset A_2 \supset ... \implies \bigcap_{i=0}^{\infty} A_i$ is infinite. This is false because of the counter-example seen in p4: $A_0 = Even \cup \{1\},\ A_1 = Even \cup \{1\} \backslash \{0\},\ A_2 = Even \cup \{1\} \backslash \{0, 2\}$,... $\bigcap_{i=0}^{\infty} A_i = 1$