Computability Assignment
Year 2012/13 - Number 2

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1 Question

In this exercise, $p(x)$ and $q(x)$ will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion);
2. $Q \subset P$ (strict inclusion);
3. $P \setminus Q \neq \emptyset$;
4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

1.1 Answer

1. Let $p(x) \Leftrightarrow x \leq 0$ and $q(x) \Leftrightarrow x \leq 1$; then $P = \{0\}$ and $Q = \{0; 1\}$.
2. This is not possible. Proof by contraddiction: Suppose such p and q exist. Then either (A)$\exists x \in \mathbb{N}.p(x)$ or (B)$\nexists x \in \mathbb{N}.p(x)$.
   If (A), then let x be any number s.t. $p(x)$. Note $x \in P \Leftrightarrow p(x)$ by def of P. So $q(x)$ holds by hypothesis, so $x \in Q$. We showed that $\forall x \in \mathbb{N}.x \in P \Rightarrow x \in Q$. So $\nexists x \in \mathbb{N}.x \in P \wedge x \notin Q$, which contraddicts the hypothesis "2.".
   If (B), then $P = \emptyset$, and there exist no Q st $Q \subset \emptyset$.

3. This is not possible. Proof by contraddiction: Suppose such p and q exist. Then either (A)$\exists x \in \mathbb{N}.p(x)$ or (B)$\nexists x \in \mathbb{N}.p(x)$.
   If (A) then, by 2.(A),$\nexists x \in \mathbb{N}.x \in P \wedge x \notin Q$, which contraddicts the hypothesis "3.".
   If (B), then $P = \emptyset$, then $P \setminus Q = \emptyset$, contraddicting "3.".

4. Let $p(x) \Leftrightarrow x \leq 0$ and $q(x) \Leftrightarrow x \leq 1$; then $P = \{0\}$ and $Q = \{0; 1\}$, so $Q \setminus P = \{1\}$.

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i \in \mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap\{A_i \mid i \in \mathbb{N}\} = \{x \mid \forall i \in \mathbb{N}\ x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap\{A_i \mid i \in \mathbb{N} \ \wedge \ i \leq k\} = A_0 \cap A_1 \cap \cdots \cap A_k$.

3 Question

Assume $(A_i)_{i \in \mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds ($\models ((*) \Rightarrow p_i)$); or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold ($\models ((*) \Rightarrow \neg p_i)$); or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i ($\models (((*) \not\Rightarrow \neg p_i) \wedge ((*) \not\Rightarrow p_i))$).

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

3.1 Answer

1. $p_1$is possible but not necessary, in fact $A_k = \bigcap_{i=0}^{k} A_i \;\Rightarrow\; A_k = A_{k-1} \cap (\bigcap_{i=0}^{k-2} A_i) \;\Rightarrow\; A_k \subseteq A_{k-1}$, but the backward implication does not hold (for example, $A_0 = \{1\}$and $\forall i \in \mathbb{N} \setminus \{0\}. A_i = \emptyset$ satisfies $(*)$ but not p_1).

2. $p_2$holds, in factthe hypothesis implies A_0finite and by$(*)$ at each step the next set is either equal to the previous (in which case we have found such j) or it has less elements. But if each set starting from A_1has less elements than the previous, then in a finite number of steps we will find $A_n = \emptyset$, so by$(*)$ it must be $A_{n+1} = \emptyset$, so we found an index that satisfies the property.

3. $p_3$is possible but not necessary: it only constraints that if there is one finite set A_i, then all subsequent sets will be equal to it, but this does not contraddict$(*)$. However,$(*)$ does not impy this (for example, $A_0 = \{1\}$and $\forall i \in \mathbb{N} \setminus \{0\}. A_i = \emptyset$ satisfies$(*)$ but not p_3).

4. $p_4$is possible but not necessary:

    (a) if $A_0 = \mathbb{N},\ A_1 = \mathbb{N} \setminus \{1\},\ A_2 = A_1 \setminus \{2\}, \ldots$ then$(*)$ and the hypothesis of p_4 hold, but $\bigcap_{i=0}^{\infty} A_i = \{0\} \neq \emptyset$, so $p_4$does not hold, so $(*) \not\Rightarrow p_4$

    (b) if $A_0 = \mathbb{N},\ A_1 = \mathbb{N} \setminus \{0\},\ A_2 = A_1 \setminus \{1\}, \ldots$ then both$(*)$ and $p_4$hold, so $(*) \not\Rightarrow \neg p_4$

5. p_5 holds, because the intersection of finite sets is always finite (as it is a subset of each finite set).

6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then$\bigcap_{i=0}^{\infty} A_i$ is finite; $p_6$is possible but not necessary:

    (a) $(\forall i \in \mathbb{N}.\ A_i = \mathbb{N})$satisfies$(*)$but not p_6, so$(*) \not\Rightarrow p_6$

    (b) $(\forall i \in \mathbb{N}.\ A_i = \{0\})$satisfies both$(*)$and p_6, so$(*) \not\Rightarrow \neg p_6$

7. $p_7$is possible but not necessary: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

    (a) $(A_0 = \mathbb{N},\ A_1 = \mathbb{N} \setminus \{1\},\ A_2 = A_1 \setminus \{2\}, \ldots)$satisfies$(*)$but not$p_7$ (because$\bigcap_{i=0}^{\infty} A_i = \{0\}$is finite), so$(*) \not\Rightarrow p_7$

    (b) $(\forall i \in \mathbb{N}.\ A_i = \{0\})$satisfies both$(*)$and p_7, so$(*) \not\Rightarrow \neg p_7$