Computability Assignment
Year 2012/13 - Number 2

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1 Question

In this exercise, $p(x)$ and $q(x)$ will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion);
2. $Q \subset P$ (strict inclusion);
3. $P \setminus Q \neq \emptyset$;
4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

1.1 Answer

1. We can define $p(x)$ as $x \leq 5$ and $q(x)$ as $x \leq 10$ therefore P is included in Q.
2. Impossible. We are dealing with natural numbers so the requirement $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ essentially says that $p(x)$ always implies $q(x)$ wich translated in sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$ means that P is a subset of Q. Therefore $Q \subset P$ (strict inclusion) is impossible.
3. From the statements above we know that $P \subset Q$ and that $Q \subset P$ is impossible. Therefore, $P \setminus Q = \emptyset$.
4. If we take the properties as defined in point 1, we have $P \subset Q$. So $Q \setminus P = \{6, 7, 8, 9, 10\} \neq \emptyset$.

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap\{A_i \mid i \in \mathbb{N}\} = \{x \mid \forall i \in \mathbb{N}\ x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap\{A_i \mid i \in \mathbb{N} \land i \leq k\} = A_0 \cap A_1 \cap \cdots \cap A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

3.1 Answer

1. p_1 holds. From the hypothesis $(*)$ we know that A_k is a subset of each of the following sets $A_0, A_1, ..., A_{k-1}$. Therefore, the intersection $A_0 \cap A_1 \cap \cdots \cap A_k = A_k$.
2. p_2 holds. It is always true because we can have two possibilities: wether $A_j = A_{j+1}$ or by decreasing, since $\forall i \in \mathbb{N}.\ A_i$ is finite we reach some point k where $A_k = A_{k+1} = \emptyset$.
3. The hypothesis is not sufficient to conclude anything about the truth of p_3.
4. p_4 holds. Since we have a strict inclusion going to infinity the intersection of the sets will be the empty set.

5. p_5 holds. $\forall i \in \mathbb{N}$. A_i is finite and the intersection of finite sets is finite.

6. p_6 does not hold. We can take $A_i = \mathbb{N}$ and the intersection is not finite.

7. p_7 holds. The hypothesis is not sufficient to conclude anything because if the sequence is decreasing then the intersection is the empty set, otherwise the intersection is infinite.