Computability Assignment
Year 2012/13 - Number 2

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1 Question

In this exercise, $p(x)$ and $q(x)$ will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion);
2. $Q \subset P$ (strict inclusion);
3. $P \setminus Q \neq \emptyset$;
4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

1.1 Answer

For P={1,3,5,7,9} and Q={1,3,5}

1. $P \subset Q$ (strict inclusion) is true because if $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ then the elements of P are also in Q;
2. $Q \subset P$ (strict inclusion) is false because if $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ then not all elements of Q are in P;
3. $P \setminus Q \neq \emptyset$ is false because since $P \subset Q$ if I remove all the elements of Q $P = \emptyset$;
4. $Q \setminus P \neq \emptyset$ is true because since $P \subset Q$ if I remove all elements of P $Q \neq \emptyset$;

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap\{A_i \mid i \in \mathbb{N}\} = \{x \mid \forall i \in \mathbb{N}\ x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cap A_1 \cap \cdots \cap A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

3.1 Answer

1. The hypothesis (*) is sufficient to conclude that p_1 holds because $A_k = A_0 \cap A_1 \cap \cdots \cap A_k$ and than for hypothesis (*) $A_k \subseteq A_{k-1} \ldots$.
2. The hypothesis (*) is sufficient to conclude that p_2 holds because if the number of elements of $A_i = n$ with $n \in \mathbb{N}$ and $n \lneq \infty$ $A_i \supset A_{i+1} \supset \ldots \supset A_{i+n}$ and when you reach the nth element $A_{i+n} = A_{i+n+1}$ and $A_{i+n} = \emptyset$.
3. The hypothesis (*) is not sufficient to conclude anything about the truth of p_3 because if A_i has 10 elements A_{i+1} may have 9 elements then $A_i \neq A_{i+1}$.
4. The hypothesis (*) is sufficient to conclude that p_4 doesn't hold because since $\exists j \in \mathbb{N}$ and $j \lneq \infty$ such that $A_j = \emptyset$for the hypothesis (*) $A_j = A_{j+1}$.

5. The hypothesis (*) is sufficient to conclude that $p_5$holds because by intersections of finite sets there is a finite set or null.

6. The hypothesis (*) is sufficient to conclude that p_6 does not hold because since $\forall A_k$ is contained in all the A_j with $0 \leq j \lneq k$ and A_k is infinite and then $\cap A_i$ is infinite.

7. The hypothesis (*) is sufficient to conclude that p_7 holds because since $\forall A_k$ is contained in all the A_j with $0 \leq j \lneq k$ and A_k is infinite and then $\cap A_i$ is infinite.