Computability Assignment
Year 2012/13 - Number 2

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1 Question

In this exercise, $p(x)$ and $q(x)$ will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion);
2. $Q \subset P$ (strict inclusion);
3. $P \setminus Q \neq \emptyset$;
4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

1.1 Answer

We can define $p(x) := 3 \leq x \leq 5$ and $q(x) := 2 \leq x \leq 6$. So It easy to see that is verified $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$.

1. $P \subset Q$ True.

$\forall x \in \mathbb{N}.\ p(x)$ we have that $x \in P \Rightarrow x \in Q$ because it's true also $q(x)$.

2. $Q \subset P$. False.

If $p(x)$ is false then $q(x)$ can be false or true. For example if $x = 6$ then $p(x)$ is false, but $q(x)$ is true.

3. $P \setminus Q \neq \emptyset$. False.

$P \subset Q$ means that every element of P is in Q, so for every element of P we can find an element of Q which "eliminate" the element of P.

4. $Q \setminus P \neq \emptyset$.True.

The strict inclusion means that Q has at least one element more than P, so Q can't "lose" all its elements.

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap\{A_i \mid i \in \mathbb{N}\} = \{x \mid \forall i \in \mathbb{N}\ x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cap A_1 \cap \cdots \cap A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

3.1 Answer

See the property:

1. True. $\bigcap_{i=0}^{k} A_i = \{x \in \mathbb{N} \,|\, (x \in A_0) \bigwedge (x \in A_1) \bigwedge \ldots \bigwedge (x \in A_k)\}$. For the hp only for $x \in A_k$ this $(x \in A_0) \bigwedge (x \in A_1) \bigwedge \ldots \bigwedge (x \in A_k)$ is true, so $\bigcap_{i=0}^{k} A_i = A_k$.

2.True. We can do an example: we have A_0 finite and we can write $\sharp A_0 = n$ and $A_0 := \{1, 2, \ldots n\}$,now for every $i+1$ we can remove an element and so $A_1 := \{1, 2 \ldots, n-1\}$ and so on, but at the end we will have $A_n = \varnothing$ and then for the hp we have to force $\forall j \in \mathbb{N} \,|\, j \geq n \; A_j = A_{j+1}$. Of course we can "stop" to remove elements when we want, so the statement is correct because we can always find a j that respect it.

3.Nothing. Doing a counterexample: we can choose $A_0 = \{1, \ldots 100\}$ and $A_1 = \{1, \ldots 99\}$. Both A_0 and $A_1$are finite and $A_1 \subset A_0$, but also we can choose all the sets equal, it respects the hp and also p_3. We need more informations.

4.False. Doing counterexample we can define $A_0 = \mathbb{N}$ and $A_i = \left\{x \in \mathbb{N} \,|\, x \, mod \, 2^i = 0\right\}$so if we do the intersection we never obtaind the empty set. In p_4 the information $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$ gives to us the information that the sets are infinite because in the other case we reach a certain j such that $A_z = \varnothing \; z \in \mathbb{N} \; z > j - 1$ and this doesn't respect the hp.

5.True. For our hp we have that the cardinality of $\bigcap_{i=0}^{\infty} A_i$ is equal to the cardinality of the smallest set, in this case the smallest set is finite and then the intersection is finite.

6.False. Doing a counterexample: $\forall i \in \mathbb{N}.\ A_i = \mathbb{N}$ and so $\bigcap_{i=0}^{\infty} A_i = \mathbb{N}$ and it is infinite.

7.True. Intersection is equal to the smallest set, but in this case it is infinite and then the intersection is infinite.