Computability Assignment
Year 2012/13 - Number 2

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1 Question

In this exercise, $p(x)$ and $q(x)$ will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion);
2. $Q \subset P$ (strict inclusion);
3. $P \setminus Q \neq \emptyset$;
4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

1.1 Answer

1. First of all I prove that 1. is equivalent to 4. and 2. is equivalent to 3., so that I will answer only at two questions.
   (a) (1. $\Rightarrow$ 4.) $P \subset Q$ means that $\forall x \in P \Rightarrow x \in Q$ and $\exists x \in Q$ such that $x \notin P$. Let's call such a x, for example, x_0. By definition $Q \setminus P = \{x | x \in Q \wedge x \notin P\}$. By constraction $Q \setminus P \neq \emptyset$because it contains at least x_0;
   (b) (4. $\Rightarrow$ 1.) $Q \setminus P = \{x | x \in Q \wedge x \notin P\} \neq \emptyset$ so $\exists x$ such that $x \in Q$ and $x \notin P$, then $P \subset Q$;
   (c) An analogous proof shows that 2. $\Leftrightarrow$ 3. .

2. Now I give an example of $p(x)$ and $q(x)$ suitable for 1. $\Leftrightarrow$ 4. .

   (a) Let's define $p(x) := (\frac{x+1}{2} \in \mathbb{N})$ and $q(x) := (x - 1 \in \mathbb{N})$;

   (b) Then I prove that $p(x) \Rightarrow q(x)$. If $p(x)$ holds means that x is odd, then x could be at least 1, from which it's obvious that $q(x)$ holds too;

   (c) Finally I prove 1. $\Leftrightarrow$ 4. . In this case, $P = \{x \in \mathbb{N} | x \equiv 1 (mod\ 2)\}$, the odd naturals, and $Q = \mathbb{N}^+ = \{n \in \mathbb{N} | n > 0\}$. $P \subset Q$ because every odd natural is greater than 0 and, for instance, $2 \in Q$ because $2 > 0$ but $2 \notin P$ because 2 is even.

3. Finally I prove the non-existence of an example suitable for 2. $\Leftrightarrow$ 3. .

   (a) Let's take $x_0 \in \mathbb{N}$ such that $x_0 \in P$ and $x_0 \notin Q$. This means that $p(x_0)$ holds and $q(x_0)$ doesn't. But this is a contraddiction to $p(x_0) \Rightarrow q(x_0)$.

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i \in \mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap \{A_i \mid i \in \mathbb{N}\} = \{x \mid \forall i \in \mathbb{N}\ x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap \{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cap A_1 \cap \cdots \cap A_k$.

3 Question

Assume $(A_i)_{i \in \mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

3.1 Answer

1. True: For any choice of $n \in \mathbb{N}$ I have that $A_{n+1} \subseteq A_n$ so that $A_{n+1} \cap A_n = A_{n+1}$. Then $\bigcap_{i=0}^{k} A_i = A_0 \cap ... \cap A_k = A_k$ since $A_k \subseteq ... \subseteq A_0$.

2. True: Let's choose $i_0 \in \mathbb{N}$ such that $A_{i_0}$is finite, that is to say $A_{i_0} = \{n_1, ..., n_t\} \subset \mathbb{N}$. Then $A_{i_0} \supseteq A_{i_0+1} \supseteq A_{i_0+2}$ and so on. They are all finite sets so that $\#A_{i_0} \geq \#A_{i_0+1}$ and so on. If there's a point in which $\#A_j = \#A_{j+1}$ I'm done, else I continue restricting the sets untill I arrive at $A_j = \{n_j\}$ with $n_j \in A_{i_0}$. Now I can't continue restricting the sets, so I'm done.

3. Unknown: Let's take $A_i = \{1, ..., i\}$, which is finite. Then A_{i+1} could be $\{1, ..., i-1\}$. It's trivial that $A_i \neq A_{i+1}$. On the other hand, if I choose $A_0 = A_1 = ... = \{0, .., n\}$, p_3 holds.

4. Unknown: If A_i are infinite is true. Intuitively, that's because the empty set is the only subset of all sets. This implies that by taking an infinite intersection of a chain of sets, the empty set is the only one which can be included in all the previous ones. But if A_i are finite sets, there are some problems. Suppose to take $A_0 = \{1, ..., n\}$, $A_1 = \{1, ..., n-1\}$and so on. I will arrive at $A_{n-1} = \{1\}$ and I have to find a proper subset of A_n, which has to be the empty set. From now, the only subset of the empty set is the empty set itself, so that starting from n, the following chain holds: $A_n = A_{n+1} = ...$. This contradicts the hypothesis of p_4.

5. True: $\bigcap_{i=0}^{\infty} A_i$ is a subset of all the A_i, which are finite. There cannot exist an infinite subset of a finite set.

6. Unknown: Let's take $\forall i \in \mathbb{N}. A_i = A_0$. In this case $\bigcap_{i=0}^{\infty} A_i = A_0$, which is infinite. Otherwise, if I take only strict subset in the chain I obtain that the intersection is the empty set, as intuitively prooved in 4.

7. Unknown: See 6.