Year 2013/14 - Number 2

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Question 1. In this exercise, $p(x)$ and $q(x)$ will be two unary proprieties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x)$ holds $\}$ and $Q = \{x \in \mathbb{N} : q(x)$ holds $\}$. If possible, for each of the cases below find two proprieties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion)
2. $Q \subset P$ (strict inclusion)
3. $P \setminus Q \neq \emptyset$
4. $Q \setminus P \neq \emptyset$

If for some of the above cases it's impossible to find such proprieties, provide a brief explanation of why is it so.

Answer 1.1.

1. Let $p(x) =$ "$\exists z \in \mathbb{N}.x = 2z + 1$" and $q(x) =$ "$x \in \mathbb{N}$". Surely, $p(x) \to q(x)$ because $z \in \mathbb{N} \to 2z+1 \in \mathbb{N} \to x \in \mathbb{N}$. Also the propriety $P \subset Q$ is satisfied: $\forall x \in P, x = 2z+1$, where $z \in \mathbb{N}$, so $x \in \mathbb{N} \to x \in Q$ but $\forall x \in Q$ such that $x = 2z$, where $z \in \mathbb{N}$, it's true that $x \notin P$ because $x = 2z \in P \to \exists y \in \mathbb{N}$ such that $2z = 2y + 1 \to 2(z - y) = 1$, where $m = z - y \in \mathbb{N}$, but the following statement "$2m = 1 \wedge m \in \mathbb{N}$" is false.
2. Using the definition of *subsets*, we obtain the following result: $Q \subset P \Leftrightarrow (\forall x \in Q \to x \in P) \leftrightarrow (\forall x.(x \in Q \to x \in P)) \leftrightarrow (\forall x.(q(x) \to p(x))$. In order to define $p(x)$ and $q(x)$ such that the propriety "$\forall x \in \mathbb{N}.p(x) \Rightarrow q(x)$" holds, it should be $P \subset Q$ (or $P \subseteq Q$). It follows that we cannot define such proprieties.

3. Recalling the previous reasoning, we can define the two proprieties iff $P \subseteq Q$. In that case, $P \setminus Q = \emptyset$.

4. Let $p(x) =$ "$x > 40$" and $q(x) =$ "$x > 30$". In that case $p(x) \rightarrow q(x)$ and $Q \setminus P = (30, 40] \neq \emptyset$.

Preliminaries 2. Given an infinite sequence of sets $(A_i)_{i \in \mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap\{A_i | i \in \mathbb{N}\} = \{x | \forall i \in \mathbb{N} x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap\{A_i | i \in \mathbb{N} \wedge i \leq k\} = A_0 \cap A_1 \cap ... \cap A_k$.

Question 3. Assume $(A_i)_{i \in \mathbb{N}}$ to be an infinite sequnce of sets of natural numbers, satisfying $\mathbb{N} \supseteq A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 ... (*)$
For each propriety p_i shown below, state whether

- the hypothesis (*) is sufficient to conclude that p_i holds; or
- the hypothesis (*) is sufficient to conclude that p_i does not hold; or
- the hypothesis (*) is not sufficient to conclude anything about the truth of p_i.

Justify your answer (briefly).

1. $p_1 : \forall k \in \mathbb{N}. A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2 : if $\forall i \in \mathbb{N}. A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3 : for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4 : if $\forall i \in \mathbb{N}. A_i \neq A_{i+1}$ then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5 : if $\forall i \in \mathbb{N}. A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6 : if $\forall i \in \mathbb{N}. A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7 : if $\forall i \in \mathbb{N}. A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

Answer 3.1.

1. p_1 holds: $\forall x \in A_k \rightarrow x \in A_m, \forall m \leq k$ such that $m, k \in \mathbb{N}$ because $A_k \subseteq A_m, \forall m \leq k$;

2. p_2 holds: suppose that $\forall i \in \mathbb{N}. A_i$ is finite and suppose that $\nexists j \in \mathbb{N}$ such that $A_j = A_{j+1}$. In that case $\forall j \in \mathbb{N}\ A_j \subset A_{j+1} \rightarrow \exists x_j \in A_j \setminus A_{j+1}$. The set $P = \{x_j | j \in \mathbb{N}\}$ is a subset of $\bigcup_{i=0}^{\infty} A_i = A_0$. There's a contraddiction because P is an infinite set (it has the power of the set of natural numbers), while A_0 is finite for hyphothesis.

3. p_3 could not hold: suppose that there are many strict inclusions, for example $A_0 = \{0, 1, 2...50\}$, $A_1 = \{1, 2....50\}$ and $A_i = \{50\} \forall i > 50$. For all i, A_i is finite, but for example $A_0 \neq A_1$. The hypothesis is not sufficient to conclude anything, because we have to make more assumptions to prove something.

4. p_4 could not hold: $\forall i \in \mathbb{N}.A_i \neq A_{i+1} \to \forall i \in \mathbb{N}.A_i \subset A_{i+1}$. If we assume that $\forall i \in \mathbb{N}\ A_i \neq \emptyset$, it follows that the intersection is not empty. The hypothesis is not sufficient to conclude anything, because we have to make more assumptions to prove something.

5. p_5 holds: $\forall j \in \mathbb{N}, \bigcap_{i=0}^{\infty} A_i \subseteq A_j \to (\forall j \in \mathbb{N}.(\forall x \in \bigcap_{i=0}^{\infty} A_i \to x \in A_j))$. The intersection is finite because every set A_j is finite.

6. p_6 does not hold: for example, suppose that $\forall i \in \mathbb{N}, A_i = A_{i+1}$. Then $A_0 = \bigcap_{i=0}^{\infty} A_i$. A_0 is infinite, so the intersection is infinite too. We can conclude this result thanks to the following proof.

7. p_7 holds: we proved that p_1 holds. It follows that $A_\infty = \bigcap_{i=0}^{\infty} A_i$. Suppose that $\forall i \in \mathbb{N}, A_i$ is infinite. So A_∞ is infinite too. It follows that the intersection above is also infinite.