Computability Assignments
Comprehensive List 2013/14

December 9, 2013

Computability Assignment
Year 2013/14 - Number 1

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More information about assignments at http://disi.unitn.it/∼zunino/teaching/computability/assignments
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1 Question

Define a binary property $p(x, y)$ over natural numbers that satisfies both the requisites:

1. $\forall x \in \mathbb{N}.\exists y \in \mathbb{N}.p(x, y)$ and
2. *it is false that* $\forall y \in \mathbb{N}.\exists x \in \mathbb{N}.p(x, y)$

Provide a definition for p, and a proof for the above claims.

1.1 Answer

Write your answer here.

Computability Assignment
Year 2012/13 - Number 2

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2 Question

In this exercise, $p(x)$ and $q(x)$ will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion);
2. $Q \subset P$ (strict inclusion);
3. $P \setminus Q \neq \emptyset$;
4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

2.1 Answer

Write your answer here.

3 Preliminaries

Given an infinite sequence of sets $(A_i)_{i \in \mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap\{A_i \mid i \in \mathbb{N}\} = \{x \mid \forall i \in \mathbb{N}\ x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cap A_1 \cap \cdots \cap A_k$.

4 Question

Assume $(A_i)_{i \in \mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

4.1 Answer

Write your answer here.

Computability Assignment
Year 2012/13 - Number 3

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5 Question

Recall the notions of image and preimage of a set with respect to a function: formally, if $A \subseteq X$, then $f(A) = \{f(x) | x \in A\} \subseteq Y$ and that, if $B \subseteq Y$, then $f^{-1}(B) = \{x | x \in X \land f(x) \in B\} \subseteq X$. (Note that here A and B are not points in the domains of f, f^{-1}, but rather sets of such points)

1. For $A \subseteq X$, determine the relation $(\subseteq, =, \supseteq)$ between A and $f^{-1}(f(A))$.
2. For $B \subseteq Y$, determine the relation $(\subseteq, =, \supseteq)$ between B and $f(f^{-1}(B))$.
3. If $C \subset A \subseteq X$, is it always true that $f(C) \subset f(A)$?
4. If $C \subset B \subseteq Y$ and $f^{-1}(B) \neq \emptyset$, is it always true that $f^{-1}(C) \subset f^{-1}(B)$?

5.1 Answer

Write your answer here.

6 Question

Let A, B be sets, and let $\mathsf{id}_A, \mathsf{id}_B$ denote the identity functions over A and B respectively. Assume $f \in (A \to B)$ and $g \in (B \to A)$ be functions satisfying $g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$, where as usual $\circ$ denotes function composition. Prove that f is a bijection (i.e., injective and surjective).

6.1 Answer

Write your answer here.

7 Question

(This question is more challenging.) Find two functions $f, g \in (\mathbb{N} \to \mathbb{N})$ that satisfy all the following conditions:

1. $\mathsf{ran}(f) \neq \mathbb{N}$ and $\mathsf{ran}(g) \neq \mathbb{N}$;
2. $\mathsf{ran}(f)$ and $\mathsf{ran}(g)$ are infinite sets;

3. $\mathsf{ran}(h) = \mathbb{N}$ where $h(n) = f(n) + g(n)$;
4. $\exists n \in \mathbb{N}.\ \mathsf{ran}(g \circ f) = \{n\}$.

7.1 Answer

Write your answer here.

Computability Assignment
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8 Question

Let A, B be sets and suppose that $A \leftrightarrow B$ (i.e. there exists a bijection $f \in (A \to B)$). Show that for all sets C, $(C \to (A \times A)) \leftrightarrow (C \to (A \times B))$.

8.1 Answer

Write your answer here.

9 Question

1. Doeas a surjective function $f \in (\mathbb{N} \to (\mathbb{N} \to \{0, 1, 2, 3\}))$ exist?
2. Does an injective function $f \in (\mathcal{P}(\mathbb{N}) \rightsquigarrow \mathbb{N})$ exist?
3. Does an injective function $f \in (\mathcal{P}(\mathbb{N}) \to \mathbb{N})$ exist?

Justify your answers.

9.1 Answer

Write your answer here.

10 Question

Let A, B be nonempty sets and let $f \in (A \to B)$. Define a function $g \in (B \rightsquigarrow A)$ such that $\mathrm{dom}(g) \neq \emptyset$ and for all $b \in \mathrm{dom(g)}$, $(f \circ g)(b) = b$.

10.1 Answer

Write your answer here.

Computability Assignment
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11 Question

Prove by induction that $\forall k \in \mathbb{N}.\, 9^k - 2^k$ is a multiple of 7. Follow the steps outlined below.

1. Prove that, if $k = 0$, then $9^0 - 2^0$ is a multiple of 7. This is the basis of the induction.

2. Now, suppose that for a *generic* natural number n, it is true that $9^n - 2^n$ is a multiple of 7. By only using this *inductive hypothesis*, prove that $9^{n+1} - 2^{n+1}$ is a multiple of 7. To do so, use the identity:

$$9^{n+1} - 2^{n+1} = 9^{n+1} - 9^n \cdot 2 + 9^n \cdot 2 - 2^{n+1}$$

   and a clever factorization of the right-hand side of the equality. Remember that, at some point, you are expected to use the *inductive hypothesis*.

11.1 Answer

Write your answer here.

12 Preliminaries

Let $P(k)$ be the property "$\forall n, m \in \mathbb{N}.\, max(n, m) = k$ implies $n = m$". The following is a proof by induction that $\forall k \in \mathbb{N}.P(k)$.

1. Basis of the induction: if $max(n, m) = 0$ then $n = m = 0$, as we wanted.

2. Inductive step: suppose that $P(k)$ is true for a generic natural number k; we want to prove that this implies $P(k + 1)$, i.e. that for all natural numbers n, m such that $max(n, m) = k + 1$, $n = m$. So, let $n, m \in \mathbb{N}$ satisfy $max(n, m) = k+1$. Then $max(n-1, m-1) = max(n, m)-1 = k$. By the induction hypothesis, it follows that $n - 1 = m - 1$, and therefore $n = m$. This proves $P(k + 1)$, so the induction step is complete.

13 Question

Is the above proof correct? If not, can you tell what is wrong with it?

13.1 Answer

Write your answer here.

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14 Question

Remember that for all $A \subseteq \mathbb{N}$, $\overline{A} = \mathbb{N} \setminus A$, and id_A is the identity function on A.
Let $f \in (\mathbb{N} \to \mathbb{N})$ and let $A = \{f(n) | n \text{ is a prime number}\}$.

1. Characterize the elements of the set $\overline{A}$ (i.e. find a property p such that $\overline{A} = \{n | p(n)\}$). Notice that p could be a conjunction of many "simpler" properties.
2. Define a function $g \in (A \to \mathbb{N})$ such that $f \circ g = \mathsf{id}_A$.

14.1 Answer

Write your answer here.

15 Question

Let $A = \{n | \exists m \in \mathbb{N}.\ n = m^2\}$ and $B = \{2n | n \in \mathbb{N}\}$. Following the steps outlined below, define a bijection $f \in (\mathbb{N} \to \mathbb{N})$ such that $f(A) = B$ and $f(\overline{A}) = \overline{B}$.

1. Provide a bijection $g \in (A \to \mathbb{N})$.
2. Provide a bijection $h \in (\mathbb{N} \to B)$.
3. Argue that there exists a bijection $g' \in (\overline{A} \to \mathbb{N})$.
4. Provide a bijection $h' \in (\mathbb{N} \to \overline{B})$.
5. Prove that the function $f \in (\mathbb{N} \to \mathbb{N})$ defined as

$$f(n) = \begin{cases} (h \circ g)(n) & if\ n \in A \\ (h' \circ g')(n) & if\ n \in \overline{A} \end{cases}$$

satisfies all the desired properties.

15.1 Answer

Write your answer here.

Computability Assignment
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16 Question

A set $C \subseteq \mathbb{N}$ is called *upward closed* iff $\forall x \in C.\ \forall y \in \mathbb{N}\ (y > x \Longrightarrow y \in C)$.

Provide a characterization of the set $Z = \{X | X \in \mathcal{P}(\mathbb{N}) \land X \text{ is upward closed}\}$ (i.e. find a property p such that $Z = \{X | X \in \mathcal{P}(\mathbb{N}) \land p(X)\}$, where p could be a conjunction of many "simpler" properties).

16.1 Answer

Write your answer here.

17 Question

A set $X \subseteq \mathbb{N}$ is called *cofinite* iff $\overline{X}$ is finite.

Prove or refute the statement: "if $X,\ Y \in \mathcal{P}(\mathbb{N})$ are NOT cofinite, then $X \cup Y$ is NOT cofinite".

17.1 Answer

Write your answer here.

18 Question

In what follows, $A \subseteq \mathbb{N}$.

1. Prove that if there exists a bijection $f \in (\mathbb{N} \to A)$, then A is infinite.

2. Can you provide an example of an infinite set A and of a function $f \in (\mathbb{N} \to A)$ which is neither injective nor surjective?

18.1 Answer

Write your answer here.

Computability Assignment
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19 Preliminaries

Recall that an equivalence relation $\sim$ over a set A is a binary relation that satisfies all of the following:

1. $\forall x \in A.\ x \sim x$ (reflexivity);
2. $\forall x, y \in A.\ x \sim y \Rightarrow y \sim x$ (symmetry);
3. $\forall x, y, z \in A.\ x \sim y \ \wedge \ y \sim z \Rightarrow x \sim z$ (transitivity).

If A is a set and $\sim$ is an equivalence relation over A, then for all $x \in A$ one can define the *equivalence class* of x with respect to $\sim$, that is the set $[x] = \{y | y \in A \ \wedge \ x \sim y\}$. We will denote by $A/\sim$ the set of all equivalence classes of elements of A, that is $A/\sim = \{[x] | x \in A\}$.

20 Question

Let A be a set and $\sim$ an equivalence relation over A. Show that, for all $x, y \in A$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$. Hint: remember that, by the *law of excluded middle*, for any choice of $x, y \in A$, either $x \sim y$ or $x \not\sim y$ (where $x \not\sim y$ means $\neg(x \sim y)$).

20.1 Answer

Write your answer here.

21 Question

Let $f \in (\mathbb{N} \to \mathbb{N})$. For each of the relations below, prove whether it is an equivalence relation over $\mathbb{N}$:

1. $x \sim y$ if and only if $f(x) = f(y)$;
2. $x \sim y$ if and only if $f(x) \neq f(y)$;
3. $x \sim y$ if and only if $f^{-1}(x) \cap f^{-1}(y) \neq \emptyset$.

21.1 Answer

Write your answer here.

22 Question

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an enumeration for the set of recursive partial functions from $\mathbb{N}$ to $\mathbb{N}$, and let $\sim$ be the equivalence relation over $\mathbb{N}$ defined as follows: $i \sim j$ if and only if $\varphi_i = \varphi_j$. Moreover, let $e \in (\mathbb{N} \times \mathbb{N} \rightsquigarrow \mathbb{N})$ the partial function defined as $e(a, b) = \varphi_a(b)$.

Prove that, if $i \sim j$, then $\forall b \in \mathbb{N},\ e(i, b) = e(j, b)$.

22.1 Answer

Write your answer here.

23 Remark

Notice that, by what you have proved in the previous exercise, it can be deduced that one can obtain a well-defined partial function $f \in (\mathbb{N}/\!\sim \times \mathbb{N} \rightsquigarrow \mathbb{N})$ by posing $f([a], b) = e(a, b)$.

Computability Assignment
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24 Definition

If $\sim$ is an equivalence relation over a set A, a set $B \subseteq A$ is closed under $\sim$ if $\forall x \in B\ \forall y \in A\ (y \sim x \Rightarrow y \in B)$.

25 Question

Let $\sim$ be the relation over $\mathbb{N}$ defined as $x \sim y$ if $|x - y|$ is a multiple of 3. Show that $\sim$ is an equivalence relation and determine all sets of natural numbers closed under $\sim$.

Hint 1: there is only a finite number of such sets.

Hint 2: take a look at question 3 below.

25.1 Answer

Write your answer here.

26 Question

Let $\sim$ be an equivalence relation over a nonempty set A. Prove that a subset $B \subseteq A$ is closed under $\sim$ if and only if it is a (possibly empty) union of equivalence classes of elements of A (for the definition of equivalence class of an element of A, see point 1 of assignment 8).

26.1 Answer

Write your answer here.

Computability Assignment
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27 Preliminaries

Recall that for $a, b \in \mathbb{N}$, $min\{a, b\}$ is the least element between a and b. Recall also that a set $C \subseteq \mathbb{N}$ is called *upward closed* iff $\forall x \in C.\ \forall y \in \mathbb{N}\ (y > x \Longrightarrow y \in C)$.

28 Question

Let $g,\ h \in \mathcal{R}$, and define

$$f(x) = \begin{cases} g(x) & \textit{whenever } x \in \mathsf{K} \\ \min\{g(x), h(x)\} + 1 & \textit{otherwise} \end{cases}$$

Is it possible to find g and h such that $f \in \mathcal{R}$ and total? If it is so, provide g, h, and the proof that $f \in \mathcal{R}$ and total; otherwise, provide a proof of why $f \notin \mathcal{R}$ or not total regardless of the choice of g and h.

28.1 Answer

Write your answer here.

29 Question

Prove or disprove: there exists an upward closed set $C \notin \mathcal{RE}$.

29.1 Answer

Write your answer here.

30 Question

Prove or disprove: the function f defined below belongs to $\mathcal{R}$.

$$f(n) = \begin{cases} (\varphi_n(n))^n & \textit{whenever } \varphi_n(n) \textit{ is defined} \\ 77 & \textit{otherwise} \end{cases}$$