Computability Final Test — 2013-09-04

Notes.

- Answer both theory questions, and choose and solve two exercises, only. Solving more exercises results in the failure of the test.
- To pass the exam you need to provide a reasonable contribution in *both* Theory and Exercises parts.
- Exercises with higher number award more points. To achieve a score ≥ 28 you have to solve an exercise marked with $\star$ below.
- Significantly wrong answers will result in negative scores.
- Always provide a justification for your answers.

Reminder: when equating the results of partial functions (as in $\phi_i(3) = \phi_i(5)$), we mean that either 1) both sides of the equation are defined, and evaluate to the same natural number, or 2) both sides are undefined.

Theory

Question 1. *Define the m-reducibility relation ($\leq_m$). Provide two sets A, B such that $A \leq_m B$ and two sets C, D such that $C \not\leq_m D$.*

Question 2. *State and prove the Rice-Shapiro theorem (part $\Leftarrow$, only).*

Exercises

Exercise 3. *Prove whether $A = \{n \mid \exists m \in \mathbb{N}.\ m \text{ even} \wedge n = m^2\}$ is $\mathcal{R}$, $\mathcal{RE} \setminus \mathcal{R}$, or not $\mathcal{RE}$.*

Solution (sketch). $A \in \mathcal{R}$. To construct a verifier for A it suffices to proceed as follows. Given an input n, compute the sequence $(2 \cdot 0)^2, (2 \cdot 1)^2, (2 \cdot 2)^2, \ldots$ until either n is found or some number $> n$ is found. In the first case, return "true", otherwise we return "false". This straightforward to implement and prove correct. □

Exercise 4. *Prove whether*

$$B = \{n \mid \phi_n(0) = \phi_n(1)\} \leq_m \mathsf{K}$$

Solution (sketch). Reduction does not hold since $B \notin \mathcal{RE}$ and $\mathsf{K} \in \mathcal{RE}$.

Indeed, we proceed by Rice-Shapiro ($\Leftarrow$). B is clearly semantically closed (easy to check), so we can take $\mathcal{F}_B$ as its associated set of partial functions. To apply Rice-Shapiro, take the finite function $g(x) = undefined$ and the recursive function $f(x) = x$. Clearly g is a restriction of f, yet $g \in \mathcal{F}_B$ ($undefined = undefined$) while $f \notin \mathcal{F}_B$ ($0 \neq 1$). □

Exercise 5. *For any two partial functions f and g, define $A_{f,g} = \{n \mid f(n) = g(n)\}$.*
Prove both the following statements:

1. *For any recursive and total f, and any recursive g, we have $A_{f,g} \in \mathcal{RE}$.*
2. *For some recursive f, and some recursive g, we have $A_{f,g} \notin \mathcal{RE}$.*

Solution (sketch).
1) We can define a semi-verifier as follows. Let `F` and `G` be any two implementations of f and g respectively. Then:

```
S(n):
    run F(n) and take its result x
    run G(n) and take its result y
    if x = y then return 1
    loop forever
```

Suppose $n \in A$, hence $f(n) = g(n)$. Since f is total, $f(n)$ is defined, and so is $g(n)$. In such case, `S(n)` correctly halts and returns 1.

Suppose $n \notin A$, hence $f(n) \neq g(n)$. Since f is total, $f(n)$ is defined. The result of $g(n)$ is either undefined or defined to some natural $\neq f(n)$. In the first case, `S(n)` will diverge when `G(n)` is run. In the second case, `S(n)` will diverge when its last line is reached.

2) Take $f(x) = undefined$ (which is $\mathcal{R}$) and $g(x) = \tilde{\chi}_{\mathsf{K}}(x)$ (which is $\mathcal{R}$). Then $A = \{n \mid undefined = \tilde{\chi}_{\mathsf{K}}(n)\} = \bar{\mathsf{K}} \notin \mathcal{RE}$.

□

Exercise 6. ⋆ *Answer the following questions:*

- *[5% score] State the second recursion theorem.*
- *[95% score] Prove that the set $A = \{n \mid \phi_n(n) = 0\}$ is <u>not</u> semantically closed.*

Solution (sketch). Intentionally omitted.

□