Computability Final Test — 2013-07-15

Notes.

- Answer both theory questions, and choose and solve two exercises, only. Solving more exercises results in the failure of the test.
- To pass the exam you need to provide a reasonable contribution in *both* Theory and Exercises parts.
- Exercises with higher number award more points. To achieve a score ≥ 28 you have to solve an exercise marked with $\star$ below.
- Significantly wrong answers will result in negative scores.
- Always provide a justification for your answers.

Reminder: when equating the results of partial functions (as in $\phi_i(3) = \phi_i(5)$), we mean that either 1) both sides of the equation are defined, and evaluate to the same natural number, or 2) both sides are undefined.

Theory

Question 1. *Prove that $A \in \mathcal{RE} \iff A \leq_m \mathsf{K}$. For each lemma used in that proof, provide its statement.*

Question 2. *State the four alternative characterizations for $\mathcal{RE}$ sets we have seen in the course (and mentioned in a single lemma in the notes). Choose two among such characterizations, and prove that implication holds between them.*

Exercises

Exercise 3. *Prove whether $\bar{\mathsf{K}} \leq_m A = \{i \mid \mathsf{ran}(\phi_i)$ infinite$\}$*

Solution (sketch). Take

$$h(n) = \# \left(\lambda x. \begin{cases} x & \text{if } \phi_n(n) \text{ does not halt in} \leq x \text{ steps} \\ undefined & \text{otherwise} \end{cases} \right)$$

The function inside the λ is partial recursive by the if-then-else lemma (on $\mathcal{R}$ conditions) since: 1) the condition "$\phi_n(n)$ does not halt in $\leq x$ steps" is recursive (implementable using the step-by-step interpreter); 2) the functions x (identity) and $undefined$ are recursive. Hence by the s-m-n lemma, h is well-defined, and recursive total.

We now check that h is indeed a reduction.

- If $n \in \bar{\mathsf{K}}$, then $\phi_n(n)$ never halts, therefore $\phi_{h(n)}(x) = x$ for all x. Hence $\mathsf{ran}(\phi_{h(n)}) = \mathbb{N}$, which is infinite. We conclude $h(n) \in B$.
- If $n \notin \bar{\mathsf{K}}$, then $\phi_n(n)$ halts in some (say k) steps. Therefore, $\phi_{h(n)}(x) = x$ for all $x < k$, and $\phi_{h(n)}(x) = undefined$ for $x \geq k$. Hence $\mathsf{ran}(\phi_{h(n)}) = \{0, \ldots, k-1\}$ which is finite. So, $h(n) \notin B$.

$\square$

Exercise 4. *Construct a binary predicate $p(x, y)$ such that we have* <u>*both*</u>*:*
1) $p(x, y)$ belongs to $\mathcal{R}$, and 2) $q(x) =$ "$\forall y.\ p(x, y)$" does not belong to $\mathcal{RE}$.
Extra point. $\star$ *Make p* <u>*also*</u> *satisfy 3) $r(x) =$ "$\exists y.\ p(x, y)$" belongs to $\mathcal{RE} \setminus \mathcal{R}$.*

Solution (sketch). Take $p(x, y) =$ "$\phi_x(0)$ does not halt in y steps". This is clearly recursive (decidable e.g. using the step-by-step interpreter). Then, $q(x)$ simplifies to "$\phi_x(0) = undefined$" which is not $\mathcal{RE}$. Indeed, a semi-verifier for q would be a semi-verifier for the set $A = \{i \mid \phi_i(0) = undefined\}$ which is not $\mathcal{RE}$ by Rice-Shapiro ($\Leftarrow$) as we now show. A is clearly semantically closed, so we let $\mathcal{F}_A$ to be its associated function set. The function $g(x) = undefined$ is a finite-domain function belonging to $\mathcal{F}_A$. Instead $f(x) = 0$ is a recursive extension of g which does not belong to $\mathcal{F}_A$.

Extra point: intentionally omitted.

$\square$

Exercise 5. *Here is an excerpt from a talk of Mr. Rouge Hareng:*

> *. . . we then obtain a set of recursive functions $\mathcal{F}$. Define $A = \{i \mid \phi_i \in \mathcal{F}\}$. We now consider two cases, according to whether the set $\mathcal{F}$ is finite or infinite.*
>
> - *If $\mathcal{F}$ is finite, from its finiteness we obtain that A is recursive.*
> - *If $\mathcal{F}$ is infinite, then A is instead not recursive.*

Comment on Mr. Hareng's claims. Provide a proof for each correct claim, and a counterexample for each wrong claim.

Solution (sketch). The first claim is wrong. If we take e.g. the finite set $\mathcal{F} = \{\mathsf{id}\}$ we have $A = \{i \mid \phi_i = \mathsf{id}\}$ which is not recursive by Rice.

The second claim is also wrong. Take e.g. $\mathcal{F} = \mathcal{R}$, which is infinite. We then obtain $A = \mathbb{N} \in \mathcal{R}$. $\square$

Exercise 6. $\star$ *Consider the following set of total functions:*

$$\mathcal{H} = \left\{ h \in (\mathbb{N}^2 \to \mathbb{N}) \;\middle|\; \forall x, n.\ \left(\begin{array}{l} x \in \mathsf{dom}(\phi_n) \implies \phi_{h(n,x)}(x) = \phi_n(x) + 1 \\ \wedge \\ x \notin \mathsf{dom}(\phi_n) \implies \phi_{h(n,x)}(x) = 0 \end{array} \right) \right\}$$

State whether $\mathcal{H} = \emptyset$. Then state whether $\mathcal{H} \cap \mathcal{R} = \emptyset$.

Solution (sketch). Intentionally omitted.

$\square$