Computability Final Test — 2013-02-04

Notes.

- Answer both theory questions, and choose and solve two exercises, only. Solving more exercises results in the failure of the test.
- To pass the exam you need to provide a reasonable contribution in *both* Theory and Exercises parts.
- Higher-numbered exercises award more points. To achieve a score ≥ 28 you have to solve the exercise marked with $\star$ below.
- Significantly wrong answers will result in negative scores.
- Always provide a justification for your answers.

Reminder: when equating the results of partial functions (as in $\phi_i(3) = \phi_i(5)$), we mean that either 1) both sides of the equation are defined, and evaluate to the same natural number, or 2) both sides are undefined.

Theory

Question 1. *State and prove the second recursion theorem for recursive functions. Then, apply it to $f(n) = \#(\lambda x.\ x^2)$, and briefly describe the result.*

Question 2. *Define the set of recursive partial functions $\mathcal{R}$. This definition involves several operators to construct a partial function from other ones. Provide a definition for these operators, too.*

Exercises

Exercise 3. *Prove whether*

$$A = \{n \mid \phi_n(0) = 7\} \quad \leq_m \quad B = \{n \mid \phi_n(7) = 0 \land \phi_n(3) \neq 0\}$$

Solution (sketch). Take

$$h(n) = \#\left(\lambda x.\ \begin{cases} 0 & \text{if } x = 7 \land \phi_n(0) = 7 \\ undefined & \text{otherwise} \end{cases}\right)$$

The predicate "$x = 7 \land \phi_n(0) = 7$" is $\mathcal{RE}$ (because ...), and 0 is a recursive function (of n, x). Hence, by the if-then-else lemma, the body inside the $(\lambda x. \ldots)$ is partial recursive. Therefore, h is well-defined and is total recursive by the s-m-n theorem.

We now check that h is a reduction:

- If $n \in A$, $\phi_n(0) = 7$. Hence, $\phi_{h(n)}(7) = 0$ and $\phi_{h(n)}(3) = undefined \neq 0$. We conclude $h(n) \in B$.

- If $n \notin A$, $\phi_n(0) \neq 7$. Hence, $\phi_{h(n)}(x) = undefined$ for all x. In particular, $\phi_{h(n)}(7) \neq 0$. We conclude $h(n) \notin B$.

$\square$

Exercise 4. *Let $A \in \mathcal{RE}$ and $B = \{b \mid \exists a \in A.\ b \leq a\}$. State which of the following properties can be concluded from these assumptions.*

1) $B \in \mathcal{R}$ 2) $B \in \mathcal{RE} \setminus \mathcal{R}$ 3) $B \notin \mathcal{RE}$ 4) *none can be concluded*

Solution (sketch). We can conclude 1) $B \in \mathcal{R}$.
We consider three exhaustive cases:

- If $A = \emptyset$, then $B = \emptyset \in \mathcal{R}$.
- If A is finite and not empty, it has a maximum element m. Then $B = \{0, \dots, m\}$ is finite, hence recursive.
- If A is infinite, $B = \mathbb{N} \in \mathcal{R}$

$\square$

Exercise 5. *Prove or refute the following statement.*
For all total functions f, g we have

$$(f \circ g) \in \mathcal{R} \implies f \in \mathcal{R} \vee g \in \mathcal{R}$$

Solution (sketch). The statement is false. Let $A = \mathsf{K} \setminus \{0, 1\}$. Then, we have $A \notin \mathcal{R}$, since otherwise $\mathsf{K} = A \cup (\mathsf{K} \cap \{0, 1\})$ would be a union of two recursive sets (the second one being finite), hence recursive.

Then, take $f = g = \chi_A \notin \mathcal{R}$. We have $(f \circ g)(x) = \chi_A(\chi_A(x)) = \chi_A(0 \text{ or } 1) = 0$. Hence $(f \circ g) \in \mathcal{R}$. $\square$

Exercise 6. $\star$ *Let $A, B \in \mathcal{RE}$. Assuming $A \cup B = \mathbb{N}$ and $A \cap B \neq \emptyset$, prove that $A \leq_m A \cap B$.*
Hint: consider $g(x) = \mu n.(\mathsf{T}(i, x, n) = 0 \vee \mathsf{T}(j, x, n) = 0)$ for suitable i, j. By construction $g(x)$ satisfies … or … . Then, define …

Solution (sketch). Intentionally omitted.

$\square$